LIMIT CYCLES IN NONLINEAR SAMPLED-DATA CONTROL SYSTEMS

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Introduction
The system shown in Fig. 1 is analysed for self sustained oscillations using the state transition method.

Two methods which are suitable for computation were developed in [1]. With the analytical method it is possible to determine whether a limit cycle of a particular u-sequence exists or not. The search method can be used on digital, analog or hybrid computer to identify different limit cycles. In this paper the two methods are further investigated and digital programs are developed for systems with general (in practical sense) piece-wise linear element [2] and general second order system. Systems of higher order can also be investigated.

Analytical method
The system shown in Fig. 1 can be represented by the vector-matrix difference equation

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\( x(k + 1) = Bx(k) + bu(k) \)

the matrix \( B \) and the vector \( b \) are functions of the sampling period \( T_s \). The method can be broadly divided into 2 categories depending on the form of the nonlinear element.

a) All the elements of the \( u \)-sequence are known:

Relay type nonlinearities fall into this category.

\( x(k + 1) = Bx(k) + bu(k) \)

\( x(k + 2) = B^2 x(k) + Bbu(k) + bu(k + 1) \)

\[ \vdots \]

\( x(k + N) = B^N x(k) + B^{N-1} bu(k) + \ldots + bu(k + N - 1) \)

For a limit cycle of period \( T \) to exist, it must be

\( x(k + N) = x(k) \quad \text{where} \quad N = \frac{T}{T_s} \)

i.e. \( (I - B^N)x(k) = [B^{N-1}u(k) + B^{N-2}u(k + 1) + \ldots + u(k + N - 1)]b \)

This gives \( x(k) \) if the matrix \( (I - B^N) \) is nonsingular

\( x(k) = (I - B^N)^{-1}[B^{N-1}u(k) + B^{N-2}u(k + 1) + \ldots + u(k + N - 1)]b \)

\( x(k + 1), x(k + 2) \ldots \text{etc. are calculated using the state transition equations and the corresponding } u \)-sequence is compared with the assumed one. Limit cycle exists if they coincide. If the matrix \( (I - B^N) \) is singular we get a band of limit cycles. For symmetrical limit cycles we use the equation

\( x(k + \frac{N}{2}) = -x(k) \)

i.e. \( (I + B^{\frac{N}{2}})x(k) = [B^{\frac{N}{2}} -1 u(k) + \ldots + u(k + \frac{N}{2} - 1)]b \)

\( x(k) = (I + B^{\frac{N}{2}})^{-1}[B^{\frac{N}{2}} -1 u(k) + \ldots + u(k + \frac{N}{2} - 1)]b \)

b) Some or all the elements of the \( u \)-sequence are not known:

From the input-output relations of the nonlinear element a
set of simultaneous equations are obtained from which $x(k)$ and all the unknown $u$-values are determined. Now starting from $x(k)$ the $u$-sequence is determined and compared with the $u$-values above to check the existence of the limit cycle.

Computation:
A computer program is made to find out the existence of limit cycles of given $u$-sequence. The program is suitable for any second order system. A piece-wise linear element of a general form is considered as the nonlinear element. By adjusting some parameters this element can be converted into most of the commonly used piecewise linear elements. The program uses the subroutine SIMQ to solve the set of simultaneous equations.

Search method
A search method can be used on digital, analog or hybrid computer to identify different limit cycles. To determine a limit cycle of definite period $T$ we begin anywhere in the state space at the initial vector $x(0)$, integrate for $T$ (or $\frac{T}{2}$ for symmetrical limit cycles), get the final vector $x(T)$, calculate $x(1)$ and take it as a new initial vector. In [1] the following recurrence formula was used:

$$x_{p+1} = \frac{1}{2}(x_p + x_{p+1})$$

The convergence was found to be slow so that it is necessary to derive other formulas with better convergence.

a) Generalization

$$x_{p+1} = x_p - \frac{f(x_p)}{R_p} \quad \text{where} \quad f(x_p) = x_p + x_{p+1}$$

In general, we can get an optimum value for $R_p$ for increased rate of convergence.

b) Newton-Raphson method (with differences):

$$x_{p+1} = x_p - \frac{f(x_p)}{f'(x_p)}$$

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here $f'(x_p)$ is approximated as

$$
\frac{(x_p + \frac{1}{x_p} - (x_{p-1} + \frac{1}{x_{p-1}})}{x_p - x_{p-1}}
$$

This approximation resulted in discriminated convergence. From
some points in the phase plane the convergence was fast where
as from others it was very slow.

c) Newton Raphson method

let $F(x_1, x_2) = 0$ and $\phi(x_1, x_2) = 0$ be the 2 equations to
be solved

$$
F(x_1, x_2) = 0 = F(x_1^p, x_2^p) + (x_1^p - x_1) F'_x + (x_2^p - x_2) F'_y + ...
$$

$$
\phi(x_1, x_2) = 0 = \phi(x_1^p, x_2^p) + (x_1^p - x_1) \phi'_x + (x_2^p - x_2) \phi'_y + ...
$$

neglecting higher derivatives we get

$$
x_1^{p+1} = x_1^p + \frac{\phi'(x_1^p, x_2^p) F_x - F'(x_1^p, x_2^p) \phi_x}{F'_x x_2^p - F'_y x_1^p}
$$

$$
x_2^{p+1} = x_2^p + \frac{\phi'(x_1^p, x_2^p) F_y - F'(x_1^p, x_2^p) \phi_y}{F'_x x_2^p - F'_y x_1^p}
$$

$F = x_1^p \pm y_1^p$, $\phi = x_2^p \pm y_2^p$

The functions $F$ and $\phi$ are determined neglecting terms indepen-
dent of the state variables ($u$ can be considered as constant).
This method was found to be powerful. It gave fast convergence in
all investigated cases.

Computation:

Similar to the analytical method a digital program was made for a
general piecewise linear element and a general second order system.
The method was also extended to systems of higher order, but the
convergence becomes slower as the order of the system increases.

Literature:

Proceedings of the 5th AICA congress, Lausanne 1967