Stability analysis and stability conditions in the delay intervals for second-order delay systems

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In this paper the stability and the stability conditions of a number of second-order retarded delay linear differential systems are discussed. The paper shows especially that the system which has stationary oscillation when without delay can become asymptotically stable in the large in certain suitable delay intervals.

1. Introduction

We shall discuss second-order retarded delay linear differential systems by means of the method mentioned in Liu and Mansour (1984). The method is based on the continuity properties of root loci with respect to delay \( h \) and eight lemmata for first-order delay differential systems in the spirit of Thowsen (1981, 1982 a, b). These fundamentals are still valid for second-order systems. But for some of the lemmata, there is a need to give the proofs once again. For the rest, however, the proofs will not be repeated here. This paper is considered as a continuation of Liu and Mansour (1984). For convenience, we shall make use of the general complex plane as well as a new coordinate system, as shown in the Figure, to describe the location of root loci and the relation of the location with delay \( h \).

In the Figure, \( h \) denotes the delay axis, \( \lambda \) the real axis and \( \omega \) the imaginary axis. The space lying on the left of the plane which is made up of \( \omega \) and \( h \) axes (\( \omega-h \)-plane) is called the left half-space (LHS), on the right it is called the right half-space (RHS).

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2. Lemmata

For the system with constant coefficients

$$
\ddot{x}(t) + a_0 x(t) + a_1 x(t-h) + \ldots + a_m x(t-mh) + b_0 \ddot{x}(t) + b_1 \ddot{x}(t-h) + \ldots + b_k \ddot{x}(t-kh) = 0
$$

(1)

where \( h > 0, m \) and \( k \) are positive integers, we have the following characteristic equation

$$
f(s) = s^2 + [a_0 + a_1 \exp(-sh) + \ldots + a_m \exp(-smh)]
$$

$$
+ s[b_0 + b_1 \exp(-sh) + \ldots + b_k \exp(-skh)] = 0
$$

i.e.

$$
f(s) = s^2 + s \sum_{q=0}^{k} b_q \exp(-qsh) + \sum_{p=0}^{m} a_p \exp(-ph) = 0
$$

(2)

Lemma 1

On the \( \omega-h \)-plane there are, in general, infinitely many intersection points of the \( \omega-h \)-plane and root loci in the delay interval \([0, \infty)\) if an intersection point exists at \( h = 0 \) for eqn. (2). The intersection points on the \( \omega-h \)-plane do not occur singly but in an infinite countable set.

Lemma 2

The intersection points on the \( \omega-h \)-plane in a set, as mentioned above in Lemma 1, have the following properties.

(a) They are symmetrical about the delay axis.

(b) They are at a distance \( \omega_i \) from the delay axis.

(c) The delay interval between two arbitrary neighbouring pairs of conjugate intersection points is \( 2\pi/\omega_i \).

(d) The signs of the real parts of the root loci derivative at the intersection points in a set are identical.

Properties (a), (b) and (c) are clear (Bellman and Cooke 1963, Hertz et al. 1984, Liu and Mansour 1984), but (d) should be proved.

Proof

When \( s = j\omega \), eqn. (2) is equivalent to the coupled equations

$$
- \omega^2 + \sum_{p=0}^{m} a_p \cos puff + \omega \sum_{q=0}^{k} b_q \sin quil = 0
$$

$$
- \sum_{p=0}^{m} a_p \sin puff + \omega \sum_{q=0}^{k} b_q \cos quil = 0
$$

Their solutions are in the form

$$
\omega_i \quad \text{and} \quad \omega_i h_n = \phi_i + 2\pi n \quad (n = \pm 0, \pm 1, \ldots)
$$

For eqn. (2) the real parts of the root locus derivative at the intersection points are

$$
\text{Re} \left( \frac{ds}{dh} \right) \bigg|_{\substack{s = j\omega \\ h = h_n}} = \frac{N}{D}
$$

(3)
where
\[ N = \omega \left( \sum_{p=0}^{m} p a_p \sin p \omega h \sum_{q=0}^{k} b_q \cos q \omega h - \sum_{p=0}^{m} p a_p \cos p \omega h \sum_{q=0}^{k} b_q \sin q \omega h \right) \]
\[ + \omega^2 \left( 2 \sum_{p=0}^{m} p a_p \cos p \omega h - \sum_{q=0}^{k} q b_q \cos q \omega h \sum_{q=0}^{k} b_q \cos q \omega h \right) \]
\[ - \sum_{q=0}^{k} q b_q \sin q \omega h \sum_{q=0}^{k} b_q \sin q \omega h \right) - 2\omega^3 \sum_{q=0}^{k} q b_q \sin q \omega h \]
\[ D = \left[ \sum_{q=0}^{k} b_q \cos q \omega h - \omega h \sum_{q=0}^{k} q b_q \sin q \omega h - h \sum_{p=0}^{m} p a_p \cos p \omega h \right]^2 \]
\[ + \left[ 2\omega - \sum_{q=0}^{k} q b_q \sin q \omega h - \omega h \sum_{q=0}^{k} q b_q \cos q \omega h + h \sum_{p=0}^{m} p a_p \sin p \omega h \right]^2 \]
Their signs are determined by the numerator \( N \) of the right-hand side of eqn. (3). Clearly, for any set of solutions \( \omega_i \) and \( \omega_i h \), the signs are identical. \( \square \)

Lemma 3
If for any constant \( h \) there is a sequence \( \{s_i\} \) of solutions of eqn. (2) such that \( |s_i| \to \infty \) as \( i \to \infty \), then \( \Re s_i \to -\infty \) as \( i \to \infty \). Thus, there is a real number \( \alpha \) such that all solutions of eqn. (2) satisfy \( \Re s < \alpha \) and there are only a finite number of solutions in any vertical strip in the complex plane.

Proof
If \( m > k \), eqn. (2) is equivalent to
\[
\left( s + b_0 + \frac{a_0}{s} \right) = - \sum_{q=1}^{k} \left( b_q + \frac{a_q}{s} \right)^{-2 \omega h} \sum_{q=1}^{m} \left( b_q + \frac{a_q}{s} \right) + \sum_{p=k+1}^{m} \exp (-p \omega h) \sum_{p=k+1}^{m} \exp (-p \omega h) \quad (4)
\]
For any solution \( s \) of eqn. (4) we have
\[
|s + b_0 + \frac{a_0}{s}| \leq \sum_{q=1}^{k} \left| \left( b_q + \frac{a_q}{s} \right) \right| (\exp (-h \Re s))^q + \sum_{p=k+1}^{m} \left| \left( \frac{a_p}{s} + 1 \right) \right| (\exp (-h \Re s))^p
\]
If \( |s| \to \infty \), then \( \exp (-h \Re s) \to \infty \), \( \Re s \to -\infty \). This implies the existence of \( \alpha \). Because \( f(s) \) is an entire function, there can be only a finite number of zeros of \( f(s) \) in any compact set. For \( k > m \) we can obtain the identical result (Hale 1977).

Lemma 4
All roots of the characteristic eqn. (2) except the two close to the solutions of the equation
\[ s^2 + \sum_{q=0}^{k} b_q s + \sum_{p=0}^{m} a_p = 0 \]
are in the left half-plane for \( h \) sufficiently small.
The lemma can be proved by the identical method mentioned in Liu and Mansour (1984) (Hale 1977).

Lemma 5

Let

\[ f(s, h) = s^2 + s \sum_{q=0}^{k} b_q \exp(-qsh) + \sum_{p=0}^{m} a_p \exp(-ps h) = 0 \]

where \( a_p \) and \( b_q \) are real numbers and \( h \geq 0 \). Then, as \( h \) varies, the sum of the multiplicities of zeros of \( f \) in the open right half-plane can change only if a zero appears on or crosses over the imaginary axis.

The lemma is quoted from Cooke and Grossman (1982).

Lemma 6

System (1) is asymptotically stable for \( h = h_p \) if and only if all root loci lie in the LHS and no locus touches the \( \omega-h \)-plane for \( h = h_p \).

Lemma 7

System (1) is asymptotically stable in the delay interval \([0, K)\) if

(a) all root loci lie in the LHS as \( h \rightarrow 0 \) (in other words

\[ \sum_{p=0}^{m} a_p > 0 \quad \text{and} \quad \sum_{q=0}^{k} b_q > 0 \]

for eqn. (2));

(b) at \( h = K \) one of the root loci crosses over (or touches) the \( \omega-h \)-plane from the LHS to the RHS for the first time as \( h \) increases.

Lemma 8

System (1) is asymptotically stable independent of delay if and only if

(a) all root loci lie in the LHS as \( h \rightarrow 0 \) (in other words

\[ \sum_{p=0}^{m} a_p > 0 \quad \text{and} \quad \sum_{q=0}^{k} b_q > 0 \]

for eqn. (2));

(b) no root locus crosses over (or touches) the \( \omega-h \)-plane for any \( h \) (Kamen 1982).

3. Stability analysis of systems and stability conditions in the delay intervals

System 1

For the system

\[ \ddot{x}(t) + a_1 x(t - h) = 0 \]  \hspace{1cm} (5)

where \( a_1 \neq 0 \), we have the characteristic equation

\[ s^3 + a_1 \exp(-sh) = 0 \]  \hspace{1cm} (6)
when \( s = j\omega \), \( a_1 > 0 \), the solutions of eqn. (6) are

\[
\omega h_n = \pm 2\pi n \quad (n = 0, 1, 2, \ldots)
\]

\[
\omega = \pm \sqrt{a_1}
\]

In view of the symmetry of the roots around the real axis, it will be sufficient to take solutions with \( \omega > 0 \) and \( \omega h_n > 0 \) (Bellman and Cooke 1963, Cooke and Grossman 1982). Correspondingly, a set of intersection points are obtained

\[
h_n = \frac{2\pi n}{\sqrt{a_1}} \quad (n = 0, 1, 2, \ldots)
\]

\[
\omega = \sqrt{a_1}
\]

This implies that the delay time \( h = 0 \) with \( s = \pm j\sqrt{a_1} \) is a pair of intersection points too.

The real parts of the derivative of the root loci at all intersection points are

\[
\text{Re} \left( \frac{ds}{dh} \right) \bigg|_{h \to h_n} = \frac{2a_1 \cos \omega h_n}{(-2\omega \sin \omega h_n - a_1 h_n)^2 + (2\omega \cos \omega h_n)^2}
\]

since \( \cos \omega h_n = 1 \), \( \text{Re} \left( \frac{ds}{dh} \right) > 0 \). This means that from \( h = 0 \) onwards, a pair of the root loci have entered in the RHS and do not go back to the LHS as \( h \to \infty \). Except at \( h = 0 \), there is at least one pair of the root loci in the RHS for any \( h \).

Clearly, for \( a_1 > 0 \) the system (5) is not asymptotically stable for any \( h \), \( \forall h \geq 0 \).

For the case of \( a_1 < 0 \), the solutions of eqn. (6) are \( \omega h_n = (2n + 1)\pi \) \( (n = 0, 1, 2, \ldots) \), \( \omega = \sqrt{-a_1} \). We have a set of intersection points

\[
h_n = \frac{(2n + 1)\pi}{\sqrt{-a_1}}
\]

\[
\omega = \sqrt{-a_1}
\]

At those intersection points the presentation of the real parts of the derivative is identical to the above. Since \( \cos \omega h_n = -1 \), and \( a_1 < 0 \), \( \text{Re} \left( \frac{ds}{dh} \right) > 0 \).

Simultaneously, eqn. (6) has a positive root \( S = \sqrt{-a_1} \) at \( h = 0 \). This implies that there is a root locus in the RHS in the sufficiently small neighbourhood of \( h = 0 \), and it remains in the RHS throughout as \( h \to \infty \). From above, the system (5) is not asymptotically stable for any \( h \), \( h \geq 0 \), no matter what the values of \( a_1 \) are

**System 2**

For the system

\[
\dot{x}(t) + a_0 x(t) + a_1 x(t-h) = 0
\]

we have the characteristic equation

\[
s^2 + a_0 + a_1 \exp(-sh) = 0
\]

For \( s = j\omega \), because complex roots occur in conjugate pairs, it will be sufficient to seek solutions with \( \omega > 0 \). Then for \( s = j\omega \) and \( a_0 > |a_1| \) we obtain two sets of its solutions

(a) \( \omega h_{1,n} = 2\pi n \) and \( \omega_2 = \sqrt{(a_0 + a_1)} \) \( (n = 0, 1, 2, \ldots) \)
Correspondingly, there are
\[ h_{1,n} = \frac{2\pi n}{\sqrt{(a_0 + a_1)}} \quad (n = 0, 1, 2, \ldots) \] (9)

\[ \omega_1 h_{1,n} = \pi + 2\pi n \quad \text{and} \quad \omega_2 = \sqrt{(a_0 - a_1)} \quad (n = 0, 1, 2, \ldots) \]

Correspondingly, there are
\[ h_{2,n} = \frac{(2n + 1)\pi}{\sqrt{(a_0 - a_1)}} \quad (n = 0, 1, 2, \ldots) \] (10)

The real parts of the derivative of the root loci at all intersection points are presented by the expression
\[ \text{Re} \left( \frac{ds}{dh} \bigg|_{h = h_n} \right) = \frac{2\omega^2 a_1 \cos \omega h_n}{(-2\omega \sin \omega h_n - a_1 h_n)^2 + (2\omega \cos \omega h_n)^2} \] (11)

We shall now discuss two cases.

**Case 1** \((a_1 > 0)\)

For \(\omega_1 h_{1,n} = 2\pi n, \cos \omega_1 h_{1,n} = 1\), the real parts of the derivative at the intersection points \((\omega_1, h_{1,n})\) are positive. Since \(\cos \omega_2 h_{2,n} = -1\), the real parts of the derivative at the points \((\omega_2, h_{2,n})\) are negative.

In the case of \(a_0 > 0, a_1 > 0\), we notice that eqn. (8) has a pair of conjugate imaginary roots for \(h = 0\). They are simultaneously the two intersection points on the \(\omega-h\)-plane, in which the real parts of the derivative are positive. From this we can see that, from there onwards, two root loci enter into the RHS, and according to Lemma 4 there are only so many root loci in the RHS in the sufficiently small neighbourhood of \(h = 0\).

Since \(a_0 + a_1 > a_0 - a_1\), from (9) and (10) the inequality \(h_{1,n} < h_{2,n}\) is valid for \(n = 0, 1, 2, \ldots\).

Based on the above and relevant lemmata, we can examine the variety of root locations as \(h\) increases. If \(h_{1,1} > h_{2,0}\), when \(0 < h < h_{2,0}\), there are two root loci in the RHS. At points \(h_{2,0} \pm j\omega_2\) the real parts of the derivative of the root loci are negative. This means that those two root loci will enter into the LHS if \(h\) continues increasing. Clearly, when \(h_{2,0} < h < h_{1,1}\), there is no root locus in the RHS, all the root loci lie in the LHS. Therefore, in this interval the system (7) is asymptotically stable in the large. At points \((h_{1,1}, \pm j\omega_1)\) the real parts of the derivative of root loci are positive. This implies that there are two root loci which will enter into the RHS, if \(h\) continues increasing, thereby the system (7) will become unstable, etc.

From the above we have seen that the system (7) possesses an alternative stability property under certain conditions.

If the inequality
\[ \left( \frac{a_0 + a_1}{a_0 - a_1} \right)^{1/2} \leq \frac{2n}{2n - 1} \] (12)
is satisfied for a certain positive integer \(n_0\), then in the delay intervals
\[ \left( \frac{(2n - 1)\pi}{\sqrt{(a_0 - a_1)}} \right) \quad \left( \frac{2\pi n}{\sqrt{(a_0 + a_1)}} \right) \]
all the root loci are in the LHS for \( n = 1, 2, \ldots, n_k \). This implies that in any such interval there is asymptotic stability in the large for system (7).

If the inequality (12) is not satisfied for \( n = 1 \), then system (7) is not asymptotically stable for any \( h \), \( \forall h \in [0, \infty) \).

**Case 2 \( (a_1 < 0) \)**

The real parts of the derivative of the root loci at the intersection points \( (\omega_1, h_{1,n}) \) are negative, but at the intersection points \( (\omega_2, h_{2,n}) \) they are positive. Besides, since \( a_0 + a_1 < a_0 - a_1 \), \( h_{1,0} = 0 \), we have

\[
h_{1,0} < h_{2,0} \quad \text{and} \quad h_{2,n} < h_{1,n+1} \quad (n = 0, 1, 2, \ldots)
\]

Clearly, in the delay interval \( (0, h_{2,0}) \) there is no root locus in the RHS. From this we can judge that the root loci, which from \( h_{2,n} \) onwards enter into the RHS, will go back separately to the LHS at \( h_{1,n+1} \) as \( h \to \infty \). Clearly, in the delay interval \( (0, h_{2,0}) \) the system is always asymptotically stable. Also, if the inequality

\[
\left( \frac{a_0 - a_1}{a_0 + a_1} \right)^{1/2} < \frac{2n + 1}{2n}
\]

(13)

(where \( n \neq 0 \)) is satisfied for a positive integer \( n_k \), then for \( n = 1, 2, \ldots, n_k \) in delay intervals

\[
\left( \frac{2\pi n}{\sqrt{a_0 + a_1}}, \frac{(2n + 1)\pi}{\sqrt{a_0 - a_1}} \right)
\]

the system (5) is asymptotically stable in the large.

Here we have seen an example of an ordinary differential system, in which there is stationary oscillation, becoming asymptotically stable after the addition of a suitable delay component.

**System 3**

For the system

\[
\dot{x}(t) + a_1 x(t-h) + a_2 x(t-2h) = 0
\]

we have the characteristic equation

\[
s^2 + a_1 \exp(-sh) + a_2 \exp(-2sh) = 0
\]

(15)

We shall now discuss two cases.

**Case 1 \( (a_2 > |a_1|) \)**

When \( s = j\omega \) from eqn. (15) we obtain two sets of solutions

(a)

\[
\omega_1 = \sqrt{a_1 + a_2} \\
\omega_1 h_{1,n} = 2\pi n \quad (n = 0, 1, 2, \ldots)
\]

(b)

\[
\omega_2 = \sqrt{a_2 - a_1} \\
\omega_2 h_{2,n} = (2n + 1)\pi \quad (n = 0, 1, 2, \ldots)
\]
and correspondingly two sets of coordinate values of the intersection points

\[
\omega_1 = \sqrt{(a_1 + a_2)} \\
h_{1,n} = \frac{2\pi n}{\sqrt{(a_1 + a_2)}} \quad (n = 0, 1, 2, \ldots)
\]

\[
\omega_2 = \sqrt{(a_2 - a_1)} \\
h_{2,n} = \frac{(2n + 1)\pi}{\sqrt{(a_2 - a_1)}} \quad (n = 0, 1, 2, \ldots)
\]

From eqn. (15) we compute \( \text{Re} (ds/dh) \) when \( s = j\omega, h = h_n \), obtaining

\[
\text{Re} \left( \frac{ds}{dh} \right) = \frac{N}{D} \quad (16)
\]

where

\[
N = 2\omega^2(a_1 \cos \omega h_n + 2a_2 \cos 2\omega h_n) \\
D = h_n^2(a_1 \cos \omega h_n + 2a_2 \cos 2\omega h_n)^2 + (2\omega + \alpha_1 h_n \sin \omega h_n + 2\alpha_2 h_n \sin 2\omega h_n)^2
\]

Consequently, the signs of \( \text{Re} (ds/dh) \) are determined by the numerator \( N \).

For \( s = j\omega_1, h_n = h_{1,n} \), since \( \omega_1 h_{1,n} = 2\pi n, \cos \omega_1 h_{1,n} = 1 = \cos 2\omega_1 h_{1,n} \), we have

\[
N = 2\omega^2(a_1 + 2a_2) > 0
\]

For \( s = j\omega_2, h_n = h_{2,n} \), since \( \omega_2 h_{2,n} = (2\pi + 1)n, \cos \omega_2 h_{2,n} = -1, \cos 2\omega_2 h_{2,n} = 1 \), we have

\[
N = 2\omega^2(2a_2 - a_1) > 0
\]

Judging from above, the root loci at all the intersection points enter into the RHS, and thereafter do not come back to the LHS as \( h \to \infty \). We should notice that \( h_{1,0} = 0 \), and so the system (14) is not asymptotically stable for any \( h \) under the condition \( a_2 > |a_1| \).

Case 2 \( (2|a_2| > a_1 > |a_2|, a_2 < 0) \)

The two sets of solutions of eqn. (15) are

\[
\omega_1 = \sqrt{(a_1 + a_2)} \\
\omega_1 h_{1,n} = 2\pi n \quad (n = 0, 1, 2, \ldots)
\]

\[
\omega_2 = \sqrt{(-a_2)} \\
\omega_2 h_{2,n} = \cos^{-1} \left( -\frac{a_1}{2a_2} \right) + 2\pi n \quad (n = 0, 1, 2, \ldots)
\]

Correspondingly, two sets of coordinate values of the intersection points are

\[
\omega_1 = \sqrt{(a_1 + a_2)} \\
h_{1,n} = \frac{2\pi n}{\sqrt{(a_1 + a_2)}} \quad (n = 0, 1, 2, \ldots)
\]
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(b) \[ \omega_2 = \sqrt{-a_2} \]
\[ h_{2,n} = \frac{\cos^{-1} \left( -\frac{a_1}{2a_2} \right) + 2\pi n}{\sqrt{-a_2}} \quad (n = 0, 1, 2, \ldots) \]

Substituting the above values into \( N \) of eqn. (16), we obtain

(a) For \( s = j\omega_1, h_{1,n} \)
\[ N_1 = 2a_2(a_1 + 2a_2) < 0 \]

(b) For \( s = j\omega_2, h_{2,n} \)
\[ N_2 = \frac{a^2(a_1^2 - 4a_2^2)}{a_2} > 0 \]

This implies that at the intersection points \((\omega_1, h_{1,n})\) the root loci tend to enter into the LHS, but at the intersection points \((\omega_2, h_{2,n})\) the root loci tend to enter into the RHS, if \( h \) increases. Since \( \omega_1 < \omega_2 \), the inequality \( h_{2,n} < h_{1,n+1} \) \((n = 0, 1, 2, \ldots)\) is guaranteed. From Lemma 4, \( N_1 < 0 \) and \( h_{1,0} = 0 \), in the delay interval \((0, h_{2,0})\) all root loci lie in the LHS. Naturally, in such a delay interval the system (14) is asymptotically stable in the large. If \( h_{1,1} < h_{2,1} \) from \( h_{2,0} \) to \( h_{1,1} \) in the RHS there is a pair of root loci, because at the intersection point \((\omega_2, h_{2,0})\) \( N_2 > 0 \), at the point \((\omega_1, h_{1,1})\) \( N_1 < 0 \), so that in the delay interval \((h_{2,0}, h_{1,1})\) the system is unstable. For the above reasons, in the delay interval \((h_{1,1}, h_{2,1})\) the system once again gains stability.

If the inequality
\[ \frac{\sqrt{(a_1 + a_2) \cos^{-1} \left( -\frac{a_1}{2a_2} \right)}}{\sqrt{(-a_2) - \sqrt{(a_1 + a_2)}}} < 2\pi n_k \]
is satisfied, then for \( n = 0, 1, \ldots, n_k \) in the delay intervals \((h_{1,n}, h_{2,n})\) the system (14) is asymptotically stable in the large.

System 4

For the system
\[ \dot{x}(t) + a_0 x(t) + b_1 x(t - h) = 0 \quad (17) \]
there is the corresponding characteristic equation
\[ s^2 + a_0 s + b_1 \exp(-sh) = 0 \quad (18) \]
In order to guarantee that for \( h = 0 \) eqn. (18) has only negative roots, \( a_0 \) and \( b_1 \) should be positive. When \( s = j\omega \), eqn. (18) is equivalent to
\[ -\omega^2 + b_1 \cos \omega h = 0 \]
\[ a_0 \omega - b_1 \sin \omega h = 0 \]
From this we obtain
\[ \omega h_n = \cos^{-1} \left( -\frac{a_0^2 + \sqrt{(a_0^4 + 4b_1^2)}}{2b_1} \right) + 2\pi n \quad (n = 0, 1, 2, \ldots) \]
\[ \omega = \left( -\frac{a_0^2 + \sqrt{(a_0^4 + 4b_1^2)}}{2} \right)^{1/2} \quad \text{(here only the positive value is taken)} \]
Here \( \omega \) and \( h_n \) are the coordinates of the intersection points where root loci intersect the \( \omega-h \)-plane. At these intersection points the real parts of the derivative of the root loci are

\[
\text{Re} \left( \frac{ds}{dh} \right) = \frac{a_0^2 \omega^2 + 2\omega^4}{(a_0 - b_1h_n \cos \omega h_n)^2 + (2\omega + b_1h_n \sin \omega h_n)^2} > 0
\]

From Lemma 4, in the case of \( a_0 > 0, b_1 > 0 \), there is no root locus in the RHS in the sufficiently small neighbourhood of \( h = 0 \). At \( h = h_0 \) a pair of the root loci cross over the \( \omega-h \)-plane from the LHS to the RHS for the first time as \( h \) increases. From above in the delay interval \( \{0, h_0 \} \) the system (17) is asymptotically stable in the large for \( a_0 > 0, b_1 > 0 \), and \( h_0 \) is the critical delay time.

**System 5**

For the system

\[
\dot{x}(t) + b_0 \dot{x}(t) + b_1 \dot{x}(t-h) + a_1 x(t-h) = 0
\]

where \( a_1 + b_0 + b_1 > 0, a_1 \neq 0 \), we have the characteristic equation

\[
s^2 + b_0 s + b_1 s \exp(-sh) + a_1 \exp(-sh) = 0
\]

It can be reformed into

\[
s^2 \exp(sh) + b_0 s \exp(sh) + b_1 s + a_1 = 0
\]

When \( s = j\omega \), eqn. (20) is equivalent to the set of equations

\[
-\omega^2 \cos \omega h - \omega b_0 \sin \omega h + a_1 = 0 \tag{21}
\]

\[
-\omega^2 \sin \omega h + \omega b_0 \cos \omega h + \omega b_1 = 0 \tag{22}
\]

From eqn. (22) we can obtain

\[
\omega = 0 \quad \text{and} \quad \omega = \frac{b_0 \cos \omega h + b_1}{\sin \omega h}
\]

Since \( a_1 \neq 0 \), from eqn. (21) \( \omega = 0 \) is invalid. Substituting \( \omega = (b_0 \cos \omega h + b_1) / \sin \omega h \) into (21), and solving eqn. (21), we obtain

\[
\cos \omega h = \frac{1}{2(b_0 b_1 + a_1)} \left( -(b_0^2 + b_1^2) \pm \sqrt{[(b_0^2 - b_1^2)^2 + 4a_1^2]} \right)
\]

To guarantee the existence of the solution, only the positive sign in the above equation is taken, and there should be

\[
|2(b_0 b_1 + a_1)| \geq \sqrt{[(b_0^2 - b_1^2)^2 + 4a_1^2]} - (b_0^2 + b_1^2)
\]

so we have

\[
\omega h = \cos^{-1} \left( \frac{-(b_0^2 + b_1^2) + \sqrt{[(b_0^2 - b_1^2)^2 + 4a_1^2]}}{2(b_0 b_1 + a_1)} \right) + 2\pi n
\]
where \( n = 0, 1, 2, \ldots \) and
\[
\omega = \frac{b_0 \cos \phi + b_1}{\sin \phi}
\]
\[
h_n = \frac{\phi + 2\pi n}{(b_0 \cos \phi + b_1) / \sin \phi} \quad (n = 0, 1, 2, \ldots)
\]
where
\[
\phi = \cos^{-1} \left( \frac{-(b_0^2 + b_1^2) + \sqrt{(b_0^2 - b_1^2)^2 + 4a_1^2}}{2(b_0b_1 + a_1)} \right)
\]
From the characteristic equation we compute \( ds/dh \). When \( s = j\omega \), its real parts are
\[
\text{Re} \left( \frac{ds}{dh} \right) = \left\{ \omega^2 \left[ \frac{1}{2} - \frac{b_0^2 - b_1^2}{2} + \sqrt{(b_0^2 - b_1^2)^2 + 4a_1^2} \right] + a_1^2 \right\} D^{-1}
\]
where
\[
D = (b_0 \cos \phi - 2\omega \sin \phi + b_1 - a_1 h_n)^2 + (2\omega \cos \phi + b_0 \sin \phi - b_1 h_n)^2
\]
Clearly, at all the intersection points they are positive. This implies that at all those intersection points the root locus crosses over the \( \omega-h \)-plane from the LHS to the RHS. Since \( a_1 + b_0 + b_1 > 0 \), \( a_1 \neq 0 \), for \( h = 0 \), eqn. (20) has only negative roots. From Lemma 4 we know that all the root loci lie in the LHS in the sufficiently small neighbourhood of \( h = 0 \), hence we can judge that in the delay interval \([0, h_0]\) all the root loci lie in the LHS, so the system (19) is asymptotically stable in the identical delay interval, and \( h_0 \) is the critical delay time.

References