Stability test and stability conditions for delay differential systems

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In this paper a stability test method is demonstrated and explicit stability conditions for the first-order retarded delay differential system with constant coefficients in the delay interval are obtained. This paper shows that under suitable conditions a class of first-order delay systems can be alternately stable as the delay $h \rightarrow \infty$.

1. Introduction

In recent years, great attention has been paid to the research into stability, independent of delay for delay differential systems, and a series of effective stability criteria and stability test methods have been presented (Kamen 1982, Lewis and Anderson 1980, Rekasius 1981, Brierly et al. 1982, Thowsen 1982, Jury and Mansour 1982). Moreover, the question of the stability in a delay interval or in several delay intervals has further been put forward (Brierly et al. 1982). It is obvious that this question is of great significance both theoretically and practically.

Because the characteristic equation of the delay differential system is an exponential polynomial equation, and equations of this kind are hard to solve, in this paper a method will be presented with which one can judge the locations of the root loci of the characteristic equation up to the stability property of the system, so that there is no need to solve the characteristic equation completely.

It should be considered that the method is a development of the idea of Thowsen (1982, 1981). The most important difference is that the distribution of the roots near the delay $h = 0$ is emphasized, and the delay $h$ is regarded as an argument.

In this paper only first-order retarded delay linear differential systems are discussed. Second-order systems can be dealt with in a similar manner and the results will be published later.

2. Theory

Consider a system of homogeneous linear differential difference equations with constant coefficients

$$\dot{x}(t) = -a_0 x(t) - a_1 x(t - h) - \ldots - a_m x(t - mh)$$  \hspace{1cm} (1)

where $m$ is a positive integer and $h > 0$.

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The characteristic equation for the system is obtained from eqn. (1) by looking for non-trivial solutions of the form $c \exp(st)$, where $c$ is constant. Equation (1) has non-trivial solutions if and only if

$$f(s) = s + a_0 + a_1 \exp(-sh) + \ldots + a_m \exp(-msh) = 0$$

(2)

According to the basic definition given by Bellman and Cooke (1963) the solution $x = \phi$ of eqn. (1) is said to be globally asymptotically stable or asymptotically stable for arbitrary perturbations, if

(a) it is stable;

(b) for all initial functions $\phi \in C([-mh, 0]; R)$, every solution $x(t, \phi)$ satisfies the relation

$$\lim_{t \to \infty} x(t, \phi) = 0$$

where $R$ is the field of real numbers.

A necessary and sufficient condition in order that the solution $x = 0$ of eqn. (1) be globally asymptotically stable is that all characteristic roots have negative real parts. In order to get stability conditions of the system in the delay interval, it is, therefore, most important to examine the inherent relations of roots of the characteristic equation to the delay $h$.

In the paper by Sugiyama (1961) the continuity properties of solutions of first-order delay differential systems with respect to the delay $h$ are proved and it is shown that the solution of the system (1) converges uniformly to the solutions of the system

$$\dot{x}(t) = - \sum_{p=0}^{m} a_p x(t)$$

(3)

as $h \to 0$.

Regarding the dependence of the roots of eqn. (2) on the delay $h$, Thowsen (1982) clearly points out that for the system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + \ldots + A_m x(t-mh)$$

(4)

where $x(t)$ is an $n$-dimensional variable, $A_0$, $A_1$, ..., $A_m$ are appropriate dimensional constant matrices, the root loci are continuous curves in the complex plane for $h \in (0, \infty)$'. Hale (1977) demonstrates that for $m=1$ all roots of the characteristic equation (2), except possibly the one close to

$$- \sum_{p=0}^{m} a_p$$

are in the left half-plane for $h$ sufficiently small. In other words, all root loci are not point-focused as $h \to 0$. This implies that constructing such a coordinate system (Fig. 1) and taking $h$ as an argument, $s$ as a function of $h$, we can get continuous space curves of root loci of eqn. (2) except at $h = 0$, if necessary.

In Fig. 1 $\omega$ denotes an imaginary axis, $\lambda$ denotes a real axis, and the negative delay axis is generally removed. The plane $s$ is made up of $\omega$ and $\omega$ axes. The space lying on the left of the plane which is made up of $\omega$ and $h$ axes ($\omega-h$-plane) is called the left half-space (LHS), on the right it is called the right half-space (RHS). We can now describe the distribution of the root loci
for eqn. (2), \( m = 1 \), in the sufficiently small neighbourhood of \( h = 0 \) as follows.

If
\[
\sum_{p=0}^{m} a_p < 0
\]
then except that a root locus is in the RHS, the rest are in the LHS; if
\[
\sum_{p=0}^{m} a_p > 0
\]
then no root locus is in the RHS, all the root loci lie in the LHS. In Lemma 4 it will be proved that the statement is still valid for \( m > 1 \). This is just the primitive condition to judge the locations of root loci as \( h \) increases.

3. Lemmata

In order to judge the locations of root loci of the characteristic equation and the stability property of systems, we need the following lemmata as well as the basic theory above.

As the delay \( h \) increases, the root loci will wander to varying degrees. It is possible that a root locus intersects the \( \omega-h \)-plane. In this connection we have Lemma 1 and 2.

**Lemma 1**

There are in general infinitely many intersection points on the \( \omega-h \)-plane in delay interval \([0, \infty)\), if a point of intersection exists at \( h_i \geq 0 \) for eqn. (2). The intersection points on the \( \omega-h \)-plane do not occur singly, but in an infinite countable set.

**Proof**

According to the premise, all points of intersection lie on the \( \omega-h \)-plane; eqn. (2) can be rewritten as

\[
j \omega a_0 + \sum_{p=1}^{m} a_p \exp(-j p \omega h) = 0
\]  \( \textbf{(5)} \)

The roots of eqn. (5) are exactly the coordinates of the intersection points.

\[
\sum_{p=1}^{m} a_p \exp(-j p \omega h)
\]
is a periodic function of $\omega h$. Let its period be $T$. If $h_i \geq 0$ satisfies eqn. (5), and eqn. (5) has a root $\omega_i$, and $\omega_i \neq 0$, then $h_n = h_i + n(T/\omega_i)$ certainly satisfies eqn. (5) too, where $\forall n \in (0, \infty)$, $n$ is an integer, and the lemma is proved. \hfill \Box

Lemma 2

The intersection points on the $\omega$-$h$ plane in a set, as mentioned above in Lemma 1, have the following properties:

(a) they are symmetrical about the delay axis;

(b) they are at a distance $\omega_i$ from the delay axis;

(c) the delay interval between two arbitrary neighbouring pairs of conjugate intersection points is $2\pi/\omega_i$;

(d) the signs of the real parts of the root loci derivative at the intersection point in a set are identical.

Proof

Eqn. (5) is equivalent to the set of equations

$$\begin{align*}
a_0 + a_1 \cos \omega h + \ldots + a_m \cos m\omega h &= 0 \\
a_1 \sin \omega h + \ldots + a_m \sin m\omega h &= 0
\end{align*}$$

If $\omega_i > 0$ and $h_i \geq 0$ are roots of the set of eqns. (6), then $\omega' = -\omega_i$ and $h_i \geq 0$ must be roots too. Actually, the solutions possess the form: $\omega = \phi + 2\pi n$ (when $\omega > 0$), or $\omega h_n = -\phi - 2\pi n$ (when $\omega < 0$) where $\phi > 0$. Clearly, the period $T$ is $2\pi$. The space coordinates of the intersection points on the $\omega$-$h$ plane in a set are

$$\omega = \pm \omega_i, \quad h_n = \frac{\phi}{\omega_i} + \frac{2\pi n}{\omega_i}$$

Then

$$\Delta h_n = h_n - h_{n-1} = \frac{2\pi}{\omega_i}$$

So (a), (b), (c) of this lemma are proved (Bellman and Cooke 1963, Hertz et al. 1984).

Cooke and Grossmann (1982) show that since eqn. (2) is an analytic function of $s$ and $h$, a root locus is a differentiable function of $h$ near $j\omega$.

From the characteristic equation (2) we have

$$\frac{ds}{dh} = \frac{sa_1 \exp (-sh) + \ldots + sma_m \exp (-smh)}{1 - (ha_1 \exp (-sh) + \ldots + hma_m \exp (-smh))}$$

At $s = j\omega$

$$\frac{ds}{dh} \bigg|_{s = j\omega} = \frac{j\omega(a_1 \exp (-j\omega h) + \ldots + ma_m \exp (-j\omega h))}{1 - (ha_1 \exp (-j\omega h) + \ldots + hma_m \exp (-j\omega h))}$$

$$\text{Re} \left( \frac{ds}{dh} \right) \bigg|_{s = j\omega} = \frac{(a_1 \sin \omega h + \ldots + ma_m \sin m\omega h)}{[1 - (h a_1 \cos \omega h + \ldots + hma_m \cos m\omega h)]^2 + h^2(a_1 \sin \omega h + \ldots + ma_m \sin m\omega h)^2}$$
Clearly, the signs of \( \text{Re} \left( \frac{ds}{dh} \right) \bigg|_{s=j\omega} \) are determined by the numerator of the right-hand side of eqn. (8). When \( s=j\omega \), from eqn. (2) we have the set of eqns. (8). For eqn. (6) there exist at the most \( m \) sets of solutions. Suppose that \( \omega_n \) and \( \omega_n h_n = \phi_i + 2\pi n \) are a set of the solutions among them, for such a set of solutions, the signs of the numerator of the right-hand side of eqn. (8) are identical.

**Lemma 3**

If there is a sequence \( \{s_i\} \) of solutions of eqn. (2) such that \( |s_i| \to \infty \) as \( i \to \infty \), then \( \text{Re} s_i \to \infty \) as \( i \to \infty \). Thus, there is a real number \( x \) such that all solutions of eqn. (2) satisfy \( \text{Re} s < x \) and there are only a finite number of solutions in any vertical strip in the complex plane.

**Proof**

The proof is similar to the one given by Hale (1677).

For any solution \( s \) of eqn. (2),

\[
|s + a_0| \leq |\alpha_1| \exp (-h \text{Re} s) + |\alpha_2| \exp (-2h \text{Re} s) + \ldots + |\alpha_m| \exp (-mh \text{Re} s)
\]

i.e.

\[
|s + a_0| \leq |\alpha_1| \exp (-h \text{Re} s) + |\alpha_2| (\exp (-h \text{Re} s))^2 + \ldots + |\alpha_m| (\exp (-h \text{Re} s))^m
\]

Clearly, if \( |s| \to \infty \), then \( \exp (-h \text{Re} s) \to \infty \) and \( \text{Re} s \to \infty \). This also implies the existence of \( x \) as in the lemma. Moreover, \( f(s) \) is an entire function. There can be only a finite number of zeros of \( f(s) \) in any compact set.

**Lemma 4**

All roots of the characteristic equation (2) except the one close to

\[- \sum_{p=0}^{m} a_p \]

are in the left half-plane for \( h \) sufficiently small.

**Proof**

For \( h \) fairly small, eqn. (2) is quite near to the equation

\[ s + \sum_{p=0}^{m} a_p [1 + p(-sh)] = 0 \]

From this we have reason to say that for any compact set \( U \) in the complex plane and any neighbourhood \( V \) of

\[- \sum_{p=0}^{m} a_p \]

there exists an \( h_\epsilon \) sufficiently small so that in \( U - V \) there is no solution of eqn. (2) and exactly one solution in \( V \) for \( 0 < h < h_\epsilon \). If \( h \to 0 \), therefore, there is
only one root which approaches

\[ - \sum_{\rho=0}^{m} a_{\rho} \]

The moduli of the other roots approach \( +\infty \). On the basis of Lemma 3 the real parts of other roots approach \( -\infty \). \qed

**Lemma 5**

Let

\[ f(s, h) = s + \sum_{\rho=0}^{m} a_{\rho} \exp(-ph) \]

where \( a_{\rho} \) are real numbers and \( h \geq 0 \). Then, as \( h \) varies, the sum of the multiplicities of zeros of \( f \) in the open right half-plane can change only if a zero appears on or crosses the imaginary axis.

**Proof**

Let \( s = s(h) \) by any root of \( f(s, h) = 0 \). If we place a small disc around \( s(h) \), then for \( h' \) sufficiently close to \( h \), the total multiplicity of roots in the disc equals the multiplicity of \( s(h) \). This follows from Rouché's theorem. In this sense, a root \( s(h) \) cannot suddenly disappear or appear or change its multiplicity at a finite point in the plane. Let \( M(h) \) be the total (finite) multiplicity of zeros in the open right half-plane. Suppose that \( M(h) \) changes but no roots appear on or cross the imaginary axis. This could only occur due to the appearance of a root at \( s = \infty \). That is, there would exist \( h^* \) and a root \( s(h) \) such that \( |s(h)| \to \infty \) as \( h \to h^* + 0 \) (or \( h \to h^* - 0 \)), with \( \text{Re} \ s(h) \geq 0 \). But this contradicts Lemma 3. \qed

Lemma 5 and its proof are basically quoted from Cooke and Grossman (1982) with minor changes.

**Lemma 6**

System (1) is asymptotically stable for \( h = h_p \), if and only if all root loci lie on the LHS and no locus touches or crosses the \( \omega-h \)-plane for \( h = h_p \).

**Proof**

This is clear from the basic theory about stability. \qed

**Lemma 7**

System (1) is asymptotically stable in the delay interval \( [0, k) \), if

(a) all root loci lie on the LHS as \( h \to 0 \), in other words,

\[ \sum_{\rho=0}^{m} a_{\rho} > 0 \]

for eqn. (2); and

(b) at \( h = k \) one of all root loci crosses over (or touches) the \( \omega-h \)-plane from the LHS to the RHS for the first time as \( h \) increases.
**Proof**

In the arbitrarily small neighbourhood of \( h = 0 \) all root loci lie on the LHS and before \( h = k \) no root locus has crossed over (or touched) the \( \omega-h \)-plane from the LHS to the RHS, according to the continuity principle. This implies that all root loci lie in the LHS in the interval \( [0, k] \), in other words, system (1) is asymptotically stable in the same interval (Lemma 6).

\[ \square \]

**Lemma 8**

System (1) is asymptotically stable independent of delay, if and only if,

(a) all root loci lie in the LHS as \( h \to 0 \), in other words

\[ \sum_{p=0}^{m} a_p > 0 \]

for eqn. (2);

(b) no root locus crosses (or touches) the \( \omega-h \)-plane for any \( h \).

**Proof**

Under the conditions mentioned above, all root loci remain throughout the LHS as \( h \to \infty \).

\[ \square \]

4. **Method**

The method will be explained by examples. For the system

\[ \dot{x}(t) + a_0 x(t) + a_1 x(t-h) = 0 \]  \( \text{(9)} \)

there exists the characteristic equation

\[ f(s, h) = s + a_0 + a_1 \exp (-sh) = 0 \]  \( \text{(10)} \)

Let \( h = 0 \), then following from eqn. (10)

\[ s = -(a_0 + a_1) \]  \( \text{(11)} \)

To get the root lying on the \( \omega-h \)-plane, let \( s = j\omega \), and from eqn. (10) it follows that

\[ s_n = j\omega \]

\[ = ja_1 \sin \left( (\pm \cos^{-1}(\frac{-a_0}{a_1}) \pm 2n\pi) \right) \]  \( \text{(12)} \)

where \( n = 0, 1, 2, \ldots \).

Keeping in mind that the intersection points are symmetric about the delay axis, formula (12) can be written as

\[ s_n = j\omega \]

\[ = ja_1 \sin \left( \cos^{-1}(\frac{-a_0}{a_1}) \pm 2\pi n \right) \]  \( \text{(13)} \)

where \( \omega \) may be positive or negative.

The coordinates of \( h \) corresponding to these roots are

\[ h_n = \frac{\cos^{-1}(\frac{-a_0}{a_1}) \pm 2\pi n}{a_1 \sin \left( \cos^{-1}(\frac{-a_0}{a_1}) \pm 2\pi n \right)} \]  \( \text{(14)} \)
In order to guarantee that \( h_n \) is positive, the signs are chosen as follows:

if \( \omega > 0, a_1 > 0 \), take \( +2\pi n \), and \( n = 0, 1, 2, \ldots \)

if \( \omega < 0, a_1 > 0 \), take \( -2\pi n \), and \( n = 1, 2, \ldots \)

Explicitly, \( h_n \) and \( s_n \) make up the space coordinates of the points of intersection of the root loci and the \( \omega-h \)-plane. At these points the root loci cross over the \( \omega-h \)-plane as \( h \) increases.

In order to judge the trends of the root loci, the derivatives of the roots with respect to \( h \) are required. In accordance with eqn. (10) they are given by

\[
\frac{ds}{dh} = \frac{a_1 s \exp(-sh)}{1 - a_1 h \exp(-sh)} \tag{15}
\]

From (13)–(15), the derivatives of all points of intersection are expressed by

\[
\left. \frac{ds}{dh} \right|_{h = h_n} = \frac{a_1^2 \sin^2 \phi - j a_1^2 \left( a_0/a_1 + (\phi/\sin \phi) \right) \sin \phi}{\left( (a_0/a_1) + (\phi/\sin \phi) \right)^2 + \sin^2 \phi} \tag{16}
\]

where

\[ \phi = \cos^{-1} \left( -a_0/a_1 \right) \pm 2\pi n \]

Now we discuss the following four cases:

**Case 1.** \( a_0 + a_1 < 0, \ a_1 \neq 0 \)

From (11) and Lemma 4, in the sufficiently small neighbourhood of \( h = 0 \) there exists a root locus which is in the RHS, the rest are in the LHS. Besides, from eqn. (16), the real parts of the derivative of all points of intersection are invariably positive. This implies that all root loci cross over the \( \omega-h \)-plane from the LHS to the RHS at these intersection points as \( h \) increases, no matter what the values of \( a_0 \) and \( a_1 \) are (except \( a_1 = 0 \) and \(|a_1| = |a_0|\)). Therefore, it is impossible that the root locus which is in the RHS will cross over the \( \omega-h \)-plane and enter the RHS as \( h \to \infty \). In the light of the continuity principle, we may reasonably conclude that this root locus remains throughout the RHS, and due to entering the RHS of new root loci one after another, there are more and more root loci in the RHS as \( h \) increases. Now we can say that the system (9) is unstable at any \( h, \forall h \in (0, \infty) \), when \( a_0 + a_1 < 0 \) and \( a_1 \neq 0 \).

**Case 2.** \( a_0 + a_1 > 0, \ a_1 > |a_0| \)

In this case where all root loci lie in the LHS in the sufficiently small neighbourhood of \( h = 0 \), and in order to guarantee that \( h_n \) are positive, eqn. (14) may be rewritten as

\[ h_n = \frac{\cos^{-1} \left( -a_0/a_1 \right) + 2\pi n}{a_1 \sin (\cos^{-1} \left( -a_0/a_1 \right) + 2\pi n)} \]

where \( n = 0, 1, 2, \ldots \)

It is quite evident that \( h_0 \) is the smallest among them. This is the first intersection point of the root loci and the \( \omega-h \)-plane, as \( h \) increases. Equation (16) gives the information that all root loci cross over the \( \omega-h \)-plane from the LHS to the RHS at these intersection points as \( h \) increases, and once the root
loci have entered the RHS they do not go back to the LHS as $h \to \infty$. According to Lemma 7, the system (9) is asymptotically stable in the interval $[0, h_0]$, and unstable for $h > h_0$. Therefore, $h_0$ may be called 'critical delay time $(h_0)$'.

Case 3. $a_0 + a_1 = 0$, $a_1 > 0$ (or $a_0 > 0$)

In this case, from (11) and Lemma 4 we know that only one root locus is close to the origin, the rest are in the LHS in the sufficiently small neighbourhood of $h = 0$. However, we do not know whether the root locus which is close to the origin is in the RHS or LHS. From (13) and (14) we have $s_n = j0$, and the coordinates of $h_n$ become

$$h_n = \begin{cases} \text{arbitrary real number}, & n = 0 \\ \infty, & n = 1, 2, 3, \ldots \end{cases}$$

From (16) we have

$$\left(\frac{ds}{dh}\right)_{h = h_n} = 0 - j0$$

Therefore, there is a root locus which lies just on the delay axis as $h$ increases and if $h \to \infty$, the multiplicity of the root loci lying on the delay axis becomes $n$. This implies that the system (9) is stationary oscillatory for every delay.

Case 4. $a_0 + a_1 > 0$, $|a_1| < |a_0|

From (11) and Lemma 4, in this case all root loci lie in the LHS in the sufficiently small neighbourhood of $h = 0$. From (13) and (14), under the existing condition ($|a_1| < |a_0|$) there is no intersection point on the $\omega$-$h$-plane. According to Lemma 8 the system (9) is globally asymptotically stable independent of delay. This condition is equivalent to $\alpha > |b|$ in Kamen (1980) and in Thowsen (1981).

We summarize the above discussions in the parametric plane. In Fig. 2 the stability properties of system (9) with different parameters are demonstrated.

5. Stability conditions in the delay interval

Let $k$ denote a given delay limitation, and $k > 0$. All of the following stability conditions are valid in the delay interval $[0, k]$ for different first-order retarded delay linear systems, if the interval is not given specifically.

1) For the system

$$\dot{x} + a_1 x(t - h) = 0$$

The stability conditions are

$$0 < a_1 \leq \frac{\pi}{2k}$$

(18)

2) For the system

$$\dot{x}(t) + a_0 x(t) + a_1 x(t - h) = 0$$
Figure 2. The stability properties of the system (9) with different parameter combinations.

The stability conditions are

(a) \[ a_1 \geq |a_0| \quad \text{(except } a_1 + a_0 = 0) \] (19)

(b) \[ \frac{\phi}{a_1 \sin \phi} \geq k \] (20)

where

\[ \phi = \cos^{-1} \left( -\frac{a_0}{a_1} \right) \] (21)

For \( a_0 > 0 \), then

\[ \frac{\pi}{2} \leq \phi \leq \pi \] (22)

\[ \frac{\pi}{2} \leq \frac{\phi}{\sin \phi} < \infty \] (23)

For \( a_0 < 0 \), then

\[ 0 \leq \phi \leq \frac{\pi}{2} \] (24)

\[ 1 \leq \frac{\phi}{\sin \phi} \leq 2 \] (25)

Example

For the system

\[ \dot{x}(t) + a_0 x(t) + 2x(t - h) = 0 \] (26)
asymptotic stability is required in the interval \([0, 1]\) and \(a_0 > 0\) is given. Determine the limits of \(a_0\). From eqn. (20) we have
\[
\frac{\cos^{-1}(-a_0/2)}{2 \sin(\cos^{-1}(-a_0/2))} \geq 1
\]
then \(a_0 \geq 0.618\), and from eqn. (19), \(a_0 \leq 2\), so the limits are \(0.618 \leq a_0 \leq 2\).

(3) For the system
\[
\dot{x}(t) + a_1 x(t-h) + a_2 x(t-2h) = 0
\]
the characteristic equation is
\[
s + a_1 \exp(-sh) + a_2 \exp(-2sh) = 0
\]
When \(s = j\omega\) and \(|a_2| > |a_1|\), it has two sets of solutions, and correspondingly there exist two sets of intersections
\[
h_{1,n} = \frac{\phi_1}{\left(\frac{a_1}{2} + \frac{1}{2} \sqrt{(a_1^2 + 8a_2^2)}\right) \sin \phi_1}
\]
where
\[
\phi_1 = \omega_1 h_{1,n} = \cos^{-1}\left(-\frac{a_1 + \sqrt{(a_1^2 + 8a_2^2)}}{4a_2}\right) + 3\pi n
\]
and \(n = 0, 1, 2, \ldots\)
\[
h_{2,n} = \frac{\phi_2}{\left(\frac{a_1}{2} - \frac{1}{2} \sqrt{(a_1^2 + 8a_2^2)}\right) \sin \phi_2}
\]
where
\[
\phi_2 = \omega_2 h_{2,n} = \cos^{-1}\left(-\frac{a_1 - \sqrt{(a_1^2 + 8a_2^2)}}{4a_2}\right) - 2\pi n
\]
and \(n = 1, 2, \ldots\).

The real parts of the derivative of the root loci at \(h_{1,n}\) are
\[
\text{Re}\left(\frac{ds}{dh}\right) \bigg|_{h=h_{1,n}} = \frac{\sin^2 \phi_1(-\frac{1}{2}a_1^2 + \frac{3}{2}a_1 \sqrt{(1 + 8(a_2/a_1)^2)} + 8a_2^2 \cos^2 \phi_1)}{h_{1,n}^2(a_1 \sin \phi_1 + 2a_2 \sin 2\phi_2)^2} > 0
\]
and at \(h_{2,n}\) are
\[
\text{Re}\left(\frac{ds}{dh}\right) \bigg|_{h=h_{2,n}} = \frac{\frac{1}{2}a_1^2 \sin^2 \phi_2 \sqrt{(1 + 8(a_2/a_1)^2)}}{h_{2,n}^2(a_1 \sin \phi_2 + 2a_2 \sin 2\phi_2)^2} > 0
\]
So far, we know that all loci cross the \(\omega-h\)-plane from the LHS to the RHS at \(h_{1,n}\) and \(h_{2,n}\).

From the above, we can conclude that for system (27) there exists a \(' critical delay time ' \(h_c\) too, if \(a_1 + a_2 > 0\). \(h_c\) is the smallest among \(h_{1,n}\) and \(h_{2,n}\). If \(h > h_c\) the system is unstable; if \(h < h_c\) the system is asymptotically stable.
So the stability conditions are

(a) \( a_2 > |a_1| \) \hspace{1cm} (35)

(b) \( h_c \geq k, \quad h_0 = \min \{h_{1,0}; h_{2,1}\} \) \hspace{1cm} (36)

Besides for \( a_1 > |a_2| \), the characteristic equation (28) has only one set of solutions, correspondingly the other stability conditions are

(a) \( \frac{a_1 + \sqrt{(a_1^2 + 8a_2^2)}}{2} \leq \frac{\phi}{k \sin \phi} \) \hspace{1cm} (37)

where

\( \phi = \cos^{-1} \left( \frac{-a_1 + \sqrt{(a_1^2 + 8a_2^2)}}{4a_2} \right) \) \hspace{1cm} (38)

(b) \( a_1 > |a_2| \) \hspace{1cm} (39)

(4) For the system

\[ \dot{x}(t) + a_2x(t) + a_1x(t-h) + a_2x(t-2h) = 0 \] \hspace{1cm} (40)

its characteristic equation generally has two sets of solutions too.

(4.1) In the case of \( a_2(a_0 - a_3) < 0 \), we can obtain two sets of intersections

\( h_{1,n} = \frac{\phi_1 + 2\pi n}{\frac{1}{2}[a_1 + \sqrt{(a_1^2 - 8a_2(a_0 - a_3))}] \sin \phi_1} \) \hspace{1cm} (41)

where

\( \phi_1 = \cos^{-1} \left( \frac{-a_1 + \sqrt{(a_1^2 - 8a_2(a_0 - a_3))}}{4a_2} \right) \) \hspace{1cm} (42)

and \( n = 0, 1, 2, \ldots \)

\( h_{2,n} = \frac{\phi_2 - 2\pi n}{\frac{1}{2}[a_1 - \sqrt{(a_1^2 - 8a_2(a_0 - a_3))}] \sin \phi_2} \) \hspace{1cm} (43)

where

\( \phi_2 = \cos^{-1} \left( \frac{-a_1 - \sqrt{(a_1^2 - 8a_2(a_0 - a_3))}}{4a_2} \right) \) \hspace{1cm} (44)

and \( n = 1, 2, \ldots \). The real parts of the derivative of the root loci at \( h_{1,n} \) and \( h_{2,n} \) are represented by eqns. (33) and (34). They are positive without exception. Clearly, for system (40) the critical delay time \( h_c \) exists, if \( a_0 + a_1 + a_2 > 0 \). The stability conditions are

(a) \( a_0 + a_1 + a_2 > 0 \) \hspace{1cm} (45)

(b) \( a_2(a_0 - a_3) < 0 \) \hspace{1cm} (46)

\[ \left| \frac{-a_1 \pm \sqrt{(a_1^2 - 8a_2(a_0 - a_3))}}{4a_2} \right| \leq 1 \] \hspace{1cm} (47)

(d) \( \begin{aligned} h_c & \geq k \\ \end{aligned} \) \hspace{1cm} (48)

where \( h_c = \min \{h_{1,0}; h_{2,1}\} \).

(4.2) In the case of \( a_2^2 > 8a_2(a_0 - a_3) > 0 \), we can obtain the following two sets of intersections

\( h_{1,n} = \frac{\phi_1 + 2\pi n}{\frac{1}{2}[a_1 + \sqrt{(a_1^2 - 8a_2(a_0 - a_3))}] \sin \phi_1} \) \hspace{1cm} (49)
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\[ h_{2,n} = \frac{\phi_1 \pm 2\pi n}{\frac{1}{2}(a_1 - \sqrt{(a_1^2 - 8a_2(a_0 - a_2))}) \sin \phi_2} \tag{50} \]

where

\[ \phi_1 \text{ is as in eqn. (42)} \]
\[ \phi_2 \text{ is as in eqn. (44)} \]

if \( a_1 > 0 \), take \( +2\pi n \), and \( n = 0, 1, 2, \ldots \)
if \( a_1 < 0 \), take \( -2\pi n \), and \( n = 1, 2, \ldots \)

For \( h_{1,n} \) the real parts of the derivative of the root loci

\[ \text{Re} \left( \frac{ds}{dh} \right)_{h=h_{1,n}} = \frac{(a_1^2/2)[1 - (8a_2(a_0 - a_2)/a_1^2)]}{[1 - h_{1,n}(a_1 \cos \phi_1 + 2a_2 \cos 2\phi_1)]^2 + h_{1,n}^2(a_1 \sin \phi_1 + 2a_2 \sin 2\phi_1)^2} + \sqrt{(1 - (8a_2(a_0 - a_2)/a_1^2))} \sin^2 \phi_1 \quad \tag{51} \]

are positive, but for \( h_{2,n} \)

\[ \text{Re} \left( \frac{ds}{dh} \right)_{h=h_{2,n}} = \frac{(a_1^2/2)[1 - (8a_2(a_0 - a_2)/a_1^2)]}{[1 - h_{2,n}(a_1 \cos \phi_2 + 2a_2 \cos 2\phi_2)]^2 + h_{2,n}^2(a_1 \sin \phi_2 + 2a_2 \sin 2\phi_2)^2} - \sqrt{(1 - (8a_2(a_0 - a_2)/a_1^2))} \sin^2 \phi_2 \quad \tag{52} \]

become negative.

Let us check the location changes of the root loci as \( h \to \infty \) under the following conditions:

\[(a) \quad a_0 + a_1 + a_2 > 0 \tag{53} \]
\[(b) \quad a_2 < 0, \quad a_1 > 0 \tag{54} \]
\[(c) \quad a_1^2 > 8a_2(a_0 - a_2) > 0 \tag{55} \]
\[(d) \quad h_{1,0} < h_{2,0} \tag{56} \]
\[(e) \quad \left| -a_1 \pm \sqrt{(a_1^2 - 8a_2(a_0 - a_2))} \right| \leq 1 \tag{57} \]

From the conditions (a), (b), (c) and (d) we have

\[ \frac{[a_1 + \sqrt{(a_1^2 - 8a_2(a_0 - a_2))}] \sin \phi_1}{[a_1 - \sqrt{(a_1^2 - 8a_2(a_0 - a_2))}] \sin \phi_2} > \frac{\phi_1}{\phi_2} > 1 \tag{58} \]

Because the interval \( \Delta h \) between the two neighbouring intersections for \( h_{1,n} \) is

\[ \frac{2\pi}{a_1 + \sqrt{(a_1^2 - 8a_2(a_0 - a_2))} \sin \phi_1} \]

and for \( h_{2,n} \)

\[ \Delta h_2 = \frac{2\pi}{a_1 - \sqrt{(a_1^2 - 8a_2(a_0 - a_2))} \sin \phi_2} \]

then we have

\[ \Delta h_2 > \Delta h_1 \tag{59} \]
and we can say that the presumed conditions guarantee \( h_{1,n} < h_{2,n} \) (for \( n = 0, 1, 2, \ldots \)). Due to \( a_0 + a_1 + a_2 > 0 \) there is no root locus on the RHS in the neighbourhood of \( h = 0 \). Judging from this, the root loci cross separately over the \( \omega - \lambda \) plane from the LHS to the RHS at \( h_{1,n} \), and afterwards go back to the LHS at \( h_{2,n} \) as \( h \rightarrow \infty \).

If there exist the conditions \( h_{2,0} < h_{1,1}, h_{2,1} < h_{1,2}, \ldots \), then we can get not only the asymptotic stability interval \([0, h_{1,1}]\), but the stability intervals \((h_{2,0}, h_{1,1}), (h_{2,1}, h_{1,2}), \ldots \) as well.

In fact, if the inequality

\[
\frac{[a_1 - \sqrt{(a_1^2 - 8a_2(a_0 - a_2))}] \sin \phi_1}{[a_1 + \sqrt{(a_1^2 - 8a_2(a_0 - a_2))}] \sin \phi_2} > \frac{\phi_1 + 2\pi n_k}{\phi_1 + 2\pi (n_k + 1)}
\]

is satisfied, then for \( n = 0, 1, \ldots, n_k \) in the delay intervals \((h_{2,n}, h_{1,n+1})\) the system (40) is globally asymptotically stable.

It is an interesting result. This means that under the conditions (3)–(5) the system (40) can be alternately stable, i.e. stable-unstable-stable-unstable as \( h \) increases.

For the system (40) the necessary and sufficient conditions of asymptotic stability independent of delay are

1. \( a_0 + a_1 + a_2 > 0 \) \hspace{1cm} (60)
2. \( a_1^2 - 8a_2(a_0 - a_2) \leq 0 \) \hspace{1cm} (61)

or

\[
\left| \frac{\sqrt{(a_1^2 - 8a_2(a_0 - a_2))} - a_1}{4a_2} \right| > 1
\] (62)

The first group of conditions can be reduced to

\( a_1^2 - 8a_2(a_0 - a_2) \leq 0 \) \hspace{1cm} (63)

and

\( a_2 > 0 \) \hspace{1cm} (64)

as it is equivalent to that given by Jury and Mansour (1982). The second group can be reduced to

\( a_2 > |a_1| + |a_2| \) \hspace{1cm} (65)

References


