ASYMPTOTIC STABILITY AND THE LYAPUNOV EQUATION FOR TWO-DIMENSIONAL DISCRETE SYSTEMS

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Abstract. The Lyapunov equation for 2-D discrete systems is investigated. In particular, the relationship between asymptotic stability, the zeros of the characteristic polynomial and the 2-D Lyapunov equation is considered. Sufficient conditions for asymptotic stability are presented based on the 2-D Lyapunov equation and the properties of quasidominant matrices.

Keywords. Stability; Lyapunov method; two-dimensional systems; discrete systems.

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INTRODUCTION

The recent interest for 2-D digital signal processing has motivated the study of the stability of 2-D discrete systems. Bounded-input bounded-output stability (BIBO stability) has been studied in many publications (see review of Jury, 1978), and since the introduction of 2-D state space models of Roesser (1975) and Kung and co-workers (1977), stability in state space representation has been considered too. Forinashin and Marchesini (1978, 1979a, 1979b, 1980); Lodge and Fahmy (1981); Kames (1978); and Pivarski (1977) have studied the concept of asymptotic stability and the 2-D Lyapunov equation. Generally, the extension of 1-D stability concepts to the 2-D case is associated with difficulties due to the increased complexity of 2-D systems. Goodman (1977) discusses these problems for systems in input output description, and Forinashin and Marchesini (1980) for systems in state space representation.

In this paper the 2-D Lyapunov equation and the relationship between asymptotic stability, zeros of the characteristic polynomial and the Lyapunov equation is considered.

The following 2-D state space model is used in this paper. The local state space is defined as the direct sum of the horizontal and vertical space, denoted by \( x^h \) and \( x^v \) respectively:

\[
\chi^h(i,j) \quad \chi^v(i,j)
\]

and

\[
\chi(i,j) = \begin{bmatrix} \chi^h(i,j) \\ \chi^v(i,j) \end{bmatrix}
\]

where \( \chi(i,j) \in \chi \) is the state and \( \chi^h(i,j) \in \chi^h \), \( \chi^v(i,j) \in \chi^v \) are the horizontal and vertical states respectively. The state space model of a 2-D discrete system is then given (Roesser, 1975; Kung and colleagues, 1977) by:

\[
\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} u(i,j)
\]

(3)

\[
\chi^h(i,j) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} = C \chi(i,j)
\]

(4)

where \( u(i,j) \) and \( \chi(i,j) \) are the input and output vectors respectively. Eq. (3) can be rewritten as

\[
\chi(i+1,j+1) = A_1 \chi(i,j) + A_2 \chi(i+1,j) + B_1 u(i+1,j) + B_2 u(i,j+1)
\]

(5)

where

\[
A_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

(6)

\[
B_1 = \begin{bmatrix} 0 \\ B_{21} \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_{11} \\ 0 \end{bmatrix}
\]

The characteristic polynomial of the state space model (3) is given by

\[
\det(z - A_1 z A_2)
\]

(7)

and the system described by (3) is asymptotically stable if

\[
\det(z - A_1 z A_2) \neq 0 \text{ in } \tilde{U}
\]

(8)

where \( \tilde{U} \) denotes the closed unit bidisc

\[
\tilde{U} = \{(z_1, z_2) | |z_1| < 1, |z_2| < 1\}
\]

(9)

Asymptotic stability can be tested by testing the zeros of the characteristic polynomial. An alternative method for testing stability is to use the 2-D Lyapunov equation. In the next section the 2-D Lyapunov equation is considered and based on it sufficient conditions for asymptotic stability are derived.
THE 2-D LYAPUNOV EQUATION

A 2-D Lyapunov equation for continuous 2-D systems was first introduced by Plekasinski (1977). However, the necessary condition for stability was not satisfactorily resolved. Lodge & Fahmy extended the continuous 2-D Lyapunov equation to the discrete case and Fornasini & Marchesini (1980) gave an alternative formulation of the discrete 2-D Lyapunov equation. In this section, the discrete 2-D Lyapunov equation is considered and simple sufficient conditions for asymptotic stability are derived for some special cases.

Theorem 1: \( \det(I-z_1A_2-z_2A_1) \neq 0 \) if there exists a block diagonal positive definite matrix \( P \) such that the matrix \( Q \), given by

\[
-Q = \begin{pmatrix} A_1^* & A_2^* \end{pmatrix}^T P \begin{pmatrix} A_1^* & A_2^* \end{pmatrix} - P
\]

\[
= \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} - \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}
\]

(10)

is positive definite.

Proof: see Humes (1978).

Theorem 1a: Given \( Q \) positive definite and \( P \) block diagonal such that (10) is satisfied, then \( \det(I-z_1A_2-z_2A_1) \neq 0 \) in \( \mathbb{C}^2 \).

Proof: Sufficiency follows from theorem 1. For necessity we will show that assuming

\[
\det(I-z_1A_2-z_2A_1) \neq 0 \text{ in } \mathbb{C}^2
\]

(11)

and \( Q \) to be positive definite in eq. (10), it follows that the block diagonal matrix \( P \) is positive definite.

From

\[
\begin{bmatrix} 1 & z_2 \end{bmatrix}^T \begin{bmatrix} 1 & -z_1 \\ 0 & 1 \end{bmatrix} z_2 = 0
\]

(12)

we obtain a nontrivial solution

\[
\begin{bmatrix} 1 & z_2 \end{bmatrix}^T \begin{bmatrix} 1 & -z_1 \\ 0 & 1 \end{bmatrix} z_2 = 0
\]

(13)

for some \((z_1, z_2) \neq (0, 0)\), and \( z_{1,1} \) are given by

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

(14)

Using the 2-D Lyapunov equation (10) we obtain

\[
\begin{align*}
\begin{bmatrix} A_1^* & A_2^* \end{bmatrix}^T P \begin{bmatrix} A_1^* & A_2^* \end{bmatrix} - P = -z_{1,0} Q_0 \\
\begin{bmatrix} z_{2,1} & 1 & 0 \\ 1 & z_{2,1} & 0 \\ 0 & 0 & 1 \end{bmatrix} (P z_{1,1} - z_{1,1} P - z_{1,1} Q) Q_0 = -z_{1,0} Q_0
\end{align*}
\]

(15)

and finally

\[
\begin{align*}
\begin{bmatrix} z_{1,1} & 1 & 0 \\ 1 & z_{1,1} & 0 \\ 0 & 0 & 1 \end{bmatrix} (P z_{2,1} - z_{2,1} P - z_{2,1} Q) Q_0 = 0
\end{align*}
\]

(16)

Due to the fact that \( Q_0 \neq 0 \) it follows that

\[
\det(I-z_1^{2-1}z_{2,1}^{2-1}P(z_1^{2-1}z_{2,1}^{2-1}P + Q)Q_0 = 0
\]

(17)

for some \((z_1, z_2) \neq (0, 0)\).

This can be considered as the characteristic equation of the following pencil (Gantmacher, 1958):

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} P - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} P + Q
\]

(18)

where

\[
\lambda = \frac{1}{1 - |z_1|^{-2}}
\]

(19)

From (11) we have that the characteristic polynomial has zeros only outside of \( \mathbb{C}^2 \) and consequently

\[
\det(I-z_1^{2-1}z_{2,1}^{2-1}P(z_1^{2-1}z_{2,1}^{2-1}P + Q)Q_0 = 0
\]

(20)

For \( |z_2| = 1 \) the pencil (18) becomes

\[
\begin{bmatrix} 0 & - \lambda \end{bmatrix} P - \lambda \begin{bmatrix} 0 & - \lambda \end{bmatrix}
\]

(21)

which is a regular pencil due to \( Q > 0 \).

Using the extremal properties of regular pencils (Gantmacher, 1958) we obtain

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} < \begin{bmatrix} 0 & - \lambda \\ - \lambda & 0 \end{bmatrix} < \lambda \begin{bmatrix} 0 & - \lambda \\ - \lambda & 0 \end{bmatrix}
\]

(22)

for all \( \lambda \in \mathbb{C} \). Using

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & z_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & z_2 \\ 0 & 1 \end{bmatrix}
\]

(23)

we have

\[
\begin{bmatrix} 1 & z_2 \\ 0 & 1 \end{bmatrix} < \begin{bmatrix} 0 & - \lambda \\ - \lambda & 0 \end{bmatrix} < \lambda \begin{bmatrix} 0 & - \lambda \\ - \lambda & 0 \end{bmatrix}
\]

(24)

\( \lambda_1 \) is given by (19) and for \( |z_2| = 1 \) follows from (20) that \( \lambda_1 > 0 \). Therefore (22) gives

\[
0 < \lambda_1 < 1 \quad \text{for all } \lambda \in \mathbb{C}
\]

(25)

Using the fact that \( Q \) is positive definite (25) implies that \( P_1 \) is positive definite too. The proof is completed by showing that \( P_2 \) is positive definite too. This can be carried out using a similar approach for the pencil given by

\[
\begin{bmatrix} 0 & - \lambda \\ - \lambda & 0 \end{bmatrix} < \begin{bmatrix} 1 & z_2 \\ 0 & 1 \end{bmatrix} < \lambda \begin{bmatrix} 0 & - \lambda \\ - \lambda & 0 \end{bmatrix}
\]

(26)

where

\[
\lambda_2 = \frac{1}{1 - |z_2|^{-2}}
\]

(27)

instead of the pencil given by (18).

Theorem 1a assumes the existence of a Q positive definite and \( P \) block diagonal such that (10) is satisfied. It is possible that for a 2-D system described by (3) having a stable characteristic polynomial, no such matrices \( P \) and \( Q \) exist. However, such systems must have a characteristic polynomial which is of higher order than 1 in both variables \( z_1 \) and \( z_2 \). For some system matrices \( A_1 \) and \( A_2 \) algebraic sufficient conditions for asymptotic stability can be given based on theorem 1 and the properties of quasi-dominant matrices (Moylan, 1977).

Theorem 2: A system described by the state space model of the form (3) is asymptotically stable if the following matrix

\[
I - \begin{bmatrix} A_1 & A_2 \end{bmatrix}
\]

has all principle minors positive. \( |A| \) denotes the matrix with elements \( a_{ij} = |a_{ij}| \).

The proof of this theorem follows directly from theorem 2 of Moylan (1977) which implies that there is a diagonal matrix \( P \) which satisfies (10). Therefore,

\[
\det(I-z_1^{2-1}z_{2,1}^{2-1}P(z_1^{2-1}z_{2,1}^{2-1}P + Q)Q_0 = 0
\]

and hence asymptotic stability follows.
Asymptotic Stability and the Lyapunov Equation

The following diagram combines the results of theorem 1 and 2:

\[ \text{asympstability} \]

\[ \text{1-D Lyapunov} \]

\[ \text{equation is} \]

\[ \text{satisfied} \]

\[ \text{I-} |A_1+A_2| \text{ has} \]

\[ \text{all principal} \]

\[ \text{minors positive} \]

The 2-D Lyapunov equation is not only useful in testing asymptotic stability but can also be used to investigate the properties of 2-D realizations. Using the 2-D Lyapunov equation, Lodge and Fahmy (1981) have shown that systems described by normal matrices \((A_1^H = A_1)\) have realizations which are free from overflow oscillations. Agathoklis, Jury and Mansour (to appear) show the same for systems described with matrices which satisfy the condition of theorem 2.

**EXAMPLES**

In this section, two examples are presented to illustrate the use of theorem 1 and 2 in testing stability of systems in state space representation.

**Example 1**

Consider the system described with the following state space model:

\[
x(n+1,n+1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix} x(n+1,n) + \begin{bmatrix} 1/3 & 0 & 1/4 \\ 1/6 & 1/2 & 1/4 \\ 0 & 0 & 0 \end{bmatrix} x(n,n+1)
\]

\[ A = A_1 + A_2 = \begin{bmatrix} 1/3 & 0 & 1/4 \\ 1/6 & 1/2 & 1/4 \\ 0 & 0 & 0 \end{bmatrix} \]

We form I-|A|

\[ I-|A| = \begin{bmatrix} 2/3 & 0 & -1/4 \\ 1/6 & 1/2 & -1/4 \\ 1/6 & 1/2 & 0 \end{bmatrix} \]

which has clearly all principal minors positive. Using theorem 2, it follows that the system (28) is asymptotically stable. The characteristic polynomial of the system is given by

\[
\det(I-xA_1 - xA_2) = \det \begin{bmatrix} 1/6 x z_1 & 0 & -1/4 x z_1 \\ -1/6 x z_1 & 1/2 z_1 & 1/4 x z_1 \\ -1/4 x z_2 & 1/2 z_2 & 1/2 x z_2 \end{bmatrix} = \]

\[ = \frac{1}{12} (2z_1)(3z_1)(2z_1) \]

which has no zeros in \( \mathbb{U}^2 \).

**Example 2**

Consider the system described by

\[
x(m+1,n+1) = \begin{bmatrix} 0 & 0 & 0 \\ 1/3 & 0 & 1/4 \\ 1/6 & 1/2 & 1/4 \\ 0 & 0 & 0 \end{bmatrix} x(m,n+1) + \begin{bmatrix} 1/6 & 1/2 & 1/4 \\ 1/6 & 1/2 & -1/4 \\ 0 & 0 & 0 \end{bmatrix} x(m,n+1)
\]

\[ I-|A_1+A_2| \text{ has no all principal minors positive} \]

In order to determine stability we choose

\[ Q = \frac{1}{2} \begin{bmatrix} 368 & 102 & 288 \\ 102 & 405 & 108 \\ 288 & 108 & 648 \end{bmatrix} \]

which give for \( P = \text{diag}(1,1,3) \). In order to obtain the matrices \( P \) and \( Q \), a set of six nonlinear inequalities with three unknowns has to be solved. The diagonal elements of \( P \) are the unknowns and the six conditions result from the condition \( P \) and \( Q \) being positive definite.

\[ P \text{ and } Q \text{ satisfy the 2-D Lyapunov equation (10)} \text{, and, therefore, the characteristic polynomial has no zeros in } \mathbb{U}^2. \text{ Indeed} \]

\[ \det(I-x_1 A_1 - x_2 A_2) = \frac{1}{6(24)^2} (96+48z_1+32z_1^2+7z_1 z_2) \]

has no zeros in \( \mathbb{U}^2 \).

**CONCLUSIONS**

The relationship between the zeros of the characteristic polynomial and the 2-D Lyapunov equation is considered. It is shown that based on the properties of quasidiagonal matrices and the 2-D Lyapunov equation sufficient conditions for asymptotic stability can be given.

**REFERENCES**


