\[
\Phi^k = \sum_{j=0}^{\infty} (AKT)^j / j!
\]  

By using (7), \( \alpha \), \( 0 \leq i < n \) can be calculated in step 3. 

Step 4, \( \Phi^k \) Step 4 gives \( L(k) \equiv CA^{-1}B/k! \) successively, and its procedure is derived as follows: let \( \omega(k) \) be the \( n \)th row vector of \( A^t \), then \( \omega(k) \) can be calculated successively as 

\[
\omega(k) = e_{n-k} - \sum_{j=0}^{k-1} a_{n-k-j} \omega(j), \quad i = k \leq n
\]

where \( e_{n-k} \) is a unit vector with the \( (n-k) \)th element one. Therefore, from (11) 

\[
L(k) = \omega(k-1) \cdot B/k!
\]

\[
= \omega(k-1) \cdot B/k!
\]

\[
= \omega(k-1) \cdot B/k!
\]

By the same manner as \( 1 \leq k \leq n \) 

\[
\omega(k) = - \sum_{j=k-n}^{k-1} a_{n-k-j} \omega(j), \quad k > n
\]

\[
L(k) = \sum_{j=k-n}^{k-1} \left( -a_{n-k-j} L(j) \right), \quad k > n
\]

\[
5^\circ \quad \Phi^i, \quad 0 \leq i < n \quad \text{are given in step 5}. \quad \text{This step is derived as follows. Using the relations (6) and (10), we can rewrite the term } \Phi^i (zI - \Phi)^{-1} \Psi A \text{ as}
\]

\[
\Phi^i (zI - \Phi)^{-1} \Psi A = \Phi^i (zI - \Phi)^{-1} (\Phi - 1)
\]

\[
= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{j=1}^{\infty} \left( a_{i+j} T^{-1} (l+1) l^{-1} / j! \right) \Phi^i
\]

As the above equation holds with respect to arbitrary \( A \) and the coefficient of \( \Phi^i (zI - \Phi)^{-1} \Psi A \) is analytic function of \( A \), then the following equation holds, and it gives step 5. 

\[
C(zI - \Phi)^{-1} \Psi B = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{j=1}^{\infty} \left( a_{i+j} T^{-1} (l+1) l^{-1} / j! \right) \Phi^i \Psi A
\]

The algorithm in single output case will be understood theoretically by the above mentioned exposition and the multoutput case will also be known by the direct expansion. At last, we add a technical comment: when executing this algorithm by a small computer, one had better adopt a little modified algorithm, which is derived by using the terms \( (T(j)^{\text{tr}} / j!) \Phi(j) \) and \( L(J)^{\text{tr}} \Phi / J! \) instead of \( (T(j)^{\text{tr}} / j!) \Phi(j) \) and \( L(J)^{\text{tr}} \Phi / J! \).

ACKNOWLEDGMENT

The author wishes to thank Prof. M. Hayase for his kind disposition to my study and Prof. E. Shimemura for his continuing guidance.

REFERENCES


A Set of Necessary and Sufficient Stability Conditions for Low Order Two-Dimensional Polynomials

E. I. JURY AND M. MANSOUR

Abstract—In this note a set of necessary and sufficient conditions for two-dimensional polynomials which are quadratic in both variables or quartic in one and linear in the other are given. In both cases the important stability condition reduces to checking the nonexistence of positive real roots of a quartic equation. Conditions for the latter have been recently presented by the authors [1]. Furthermore, by appealing to coordinate transformation, the explicit stability conditions are extended to the largest possible class of two-dimensional polynomials. Finally, a sufficient condition for stability is given for any two-dimensional polynomial.

I. INTRODUCTION

In recent years, we notice considerable research activity in two and multidimensional digital filters. In particular, problems related to stability [2], and to design are being vigorously pursued. It is often required to obtain a set of necessary and sufficient conditions for the stability of two-dimensional recursive and causal filters. In this note, we present a class of two-dimensional filters for which the stability conditions are given in terms of coefficients of the appropriate two-dimensional polynomials. This set represents the largest class of two-dimensional polynomials for which the stability conditions have been given explicitly. It reduces to checking the positivity of two-dimensional quartic polynomials. The test for the latter was recently given by the authors [1]. The discussion follows that of a recently published paper [3].

II. MATHEMATICAL FORMULATION

Let the denominator of a two-dimensional recursive filter be given in either one of the two forms.

1) \( B(z_1, z_2) = \left( h_{00} + h_{10} z_1 + h_{20} z_1^2 + h_{01} z_2^2 + h_{11} z_1^2 z_2 + h_{21} z_1 z_2^2 + h_{02} z_2^4 \right) \)

\[ + \left( h_{01} + h_{11} z_1^2 + h_{21} z_1^2 z_2 + h_{12} z_1^3 z_2 + h_{22} z_2^4 \right) z_2 \]

Equation 1) is linear in \( z_2 \) and quartic in \( z_1 \) 

2) \( B(z_1, z_2) = \left( h_{00} + h_{10} z_1 + h_{20} z_1^2 + h_{01} z_2^2 + h_{11} z_1^2 z_2 + h_{21} z_1 z_2^2 + h_{02} z_2^4 \right) z_2 \)

\[ + \left( h_{01} + h_{11} z_1^2 + h_{21} z_1^2 z_2 + h_{12} z_1^3 z_2 + h_{22} z_2^4 \right) z_2 \]

Equation 2) is quadratic in both variables.

For checking positivity at one point reduces to a one-dimensional polynomial problem and hence it is straight-forward. The checking of (4) reduces to checking the positiveness of the Schur-Cohn-Hermitian matrix. The latter condition is satisfied, if it is positive at one point and the determinant \( \det H(z_1) = f(z_1) \) is positive for all \( |z_1| = 1 \). Checking positivity at one point reduces to a constant matrix being positive. Now the determinant is a function of \( (z_1 + z_1^{-1}) \), for on the unit circle \( z_1 z_1^{-1} = \bar{z}_1 \) where the star denotes complex conjugation. Thus, by substituting the bilinear transformation for 

\[
z_1 = \frac{1 + e^{j\theta}}{1 - e^{j\theta}} \quad \text{where } s = \frac{1 + e^{j\theta}}{1 - e^{j\theta}}
\]

The checking of (3) is a one-dimensional polynomial problem and hence it is straightforward. The checking of (4) reduces to checking the positiveness of the Schur-Cohn-Hermitian matrix. The latter condition is satisfied, if it is positive at one point and the determinant \( \det H(z_1) = f(z_1) \) is positive for all \( |z_1| = 1 \). Checking positivity at one point reduces to a constant matrix being positive. Now the determinant is a function of \( (z_1 + z_1^{-1}) \), for on the unit circle \( z_1 z_1^{-1} = \bar{z}_1 \) where the star denotes complex conjugation. Thus, by substituting the bilinear transformation for

\[
z_1 = \frac{1 + e^{j\theta}}{1 - e^{j\theta}} \quad \text{where } s = \frac{1 + e^{j\theta}}{1 - e^{j\theta}}
\]
This in turn requires that there exist no positive real roots. All other conditions for stability are straightforward to test.

**Remark 1:** One can also obtain a sufficient condition for stability of any two-dimensional polynomial, by noting the sufficiency condition of positivity for any general polynomial \(f(y)\) \[1\]. This condition is given by the positivity of the coefficients of \(f(y)\).

**Remark 2:** Using coordinate transformation, we can enlarge the class of two-dimensional polynomials for stability \[4\]. Indeed, if \(B(z_1, z_2)\) is stable, then \(B(z_1 z_2^{k_3}, z_1^{k_1} z_2^{k_2})\) is also stable where \(k_1, k_2, k_3\) are non-negative integers.

### Conclusion

In this note we obtained the stability conditions of a larger class of two-dimensional polynomials than any other previously obtained. The stability condition reduces to the positivity condition of a quartic equation. Furthermore, we obtained a sufficient condition for stability for any two-dimensional polynomials. The sufficiency condition reduces to the positivity of the coefficients of the transformed polynomial in the discussion. Application of this note to the design of two-dimensional digital filters with guaranteed stability is straightforward.

### Acknowledgment

The authors are grateful to Prof. N. K. Bose for his constructive comments on this note, and to B. Cagatay for his aid in the derivations.

### References


### On the Cayley–Hamilton Theorem for Two-Dimensional Systems

**TURHAN ÇİFTÇİBAŞI AND ÖNDER YÜKSEL**

**Abstract**—An alternate proof of the 2-D Cayley–Hamilton theorem is supplied together with an extension to derived polynomials.

Two-dimensional extension of the well-known Cayley–Hamilton theorem was first given by Givone and Roesser [1] and a different proof was supplied by Vilfan [2]. In this note an alternate proof, which also proves that the derived polynomials too are satisfied by matrix \(A\), is given.

Consider the square, partitioned matrix \(A\) as

\[
A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}
\]

(1)

where \(A_1, A_2, A_3, A_4\) are \(n \times n, n \times m, m \times n, \) and \(m \times m\) matrices, respectively. The matrices \(A^{-j}\) are defined inductively as

\[
A^{1,0} = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}, \quad A^{0,1} = \begin{bmatrix} 0 & 0 \\ A_3 & A_4 \end{bmatrix}
\]

(2a)

\[
A^{-j} = 0 \quad \text{for either of } i < 0 \quad \text{or} \quad j < 0,
\]

(2b)

\[
A^{-j} = A^{1,0} A^{1,-j} + A^{0,1} A^{j,-1} \quad \text{for } i > 0, j > 0.
\]

(2c)

Manuscript received February 11, 1981; revised April 2, 1981.

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0018-9286/82/0200-0193$00.75 ©1982 IEEE