A Note on the Stability of Linear Discrete Systems and Lyapunov Method

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Abstract — In a publication by Y. P. Ham and C. T. Chen [1] a proof of a discrete stability test by using the Lyapunov method was given. In this note the main results obtained by the author in this direction in 1965 [6], [7] and the extensions and applications obtained partially with other authors since then and published in different journals [8]–[10] are surveyed. The similarity to the results in [1] is obvious.

I. INTRODUCTION

Wall [2] has shown that the characteristic polynomial of a continuous system \( p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n \) can be represented by the matrix

\[
F(s) = \begin{bmatrix}
    b_1 + s & -b_2 & & & -b_n \\
     1 & -b_3 & & & & & \\
     & 1 & -b_4 & & & & \\
     & & \ddots & \ddots & & & \\
     & & & 1 & -b_n \\
    \end{bmatrix}
\]

i.e., the roots of \( p(s) \) are the eigenvalues of the matrix

\[
S = \begin{bmatrix}
    -b_1 & -b_2 & & \cdots & -b_n \\
     1 & 0 & & & & & \\
     & 1 & & \ddots & & \cdots & \\
     & & \ddots & \ddots & & & \cdots & \\
     & & & 1 & 0 \\
    \end{bmatrix}
\]

which was later known as the Schwarz matrix. Schwarz [3] has given a numerical method to transform a system matrix to this form for stability investigation. In case of stability \( b_1, b_2, \ldots, b_n \) should be positive. Kalman and Bertram [4] used a quadratic form with a diagonal matrix to prove this last result of Schwarz by Lyapunov method. Parks [5] established the relation between the \( b \)'s and Hurwitz determinants. This closed the link between Routh–Hurwitz criterion and Lyapunov method for continuous systems. It was pointed out by Kalman and Bertram [4] that the analog of the canonic matrix \( S \) is not available for discrete systems. Mansour [6], [7] developed the analogous results. The resulting matrix for discrete systems has the following form:

\[
H_1 = \begin{bmatrix}
    -\Delta_{n-1} \Delta_n & 1 - \Delta_{n-1}^2 & 0 & & & 0 \\
    -\Delta_{n-2} \Delta_n & -\Delta_{n-3} \Delta_{n-1} & -1 - \Delta_{n-2}^2 \\
    -\Delta_0 \Delta_n & -\Delta_{n-1} \Delta_{n-1} & -\Delta_{n-2} \Delta_{n-2} & -1 - \Delta_1^2 \\
    \end{bmatrix}
\]

or

\[
H_2 = \begin{bmatrix}
    -\Delta_{n-1} \Delta_n & 1 \\
    -\Delta_{n-2} \Delta_n (1 - \Delta_{n-1}^2) & -\Delta_{n-2} \Delta_{n-1} \\
    -\Delta_{n-3} \Delta_n (1 - \Delta_{n-2}^2) (1 - \Delta_{n-1}^2) & -\Delta_{n-3} \Delta_{n-2} \Delta_{n-1} (1 - \Delta_{n-2}^2) \\
    \end{bmatrix}
\]

Another form can be obtained using the transformation matrix \( T_2 \) [9] whose elements are given by

\[
F_d(z) = z^n + a_1 z^{n-1} + \cdots + a_n = 0.
\]

The necessary and sufficient conditions for the roots of (2) to lie inside the unit circle are that the zeroth order terms in \( F_d(z) \) and the \( n-1 \) polynomials obtained successively through the transformation

\[
F_{r-1}(z) = \frac{1}{z^r (1 - \Delta_r^2)} F_r(z) - \Delta_r z F_1 \left( \frac{1}{z} \right)
\]

have magnitude smaller than unity. \( \Delta_r \) is the zeroth order term of the polynomial of degree \( r \)

\[
|\Delta_r| < 1, \quad r = 1, 2, \ldots, n.
\]

This is one form of the Schur--Cohn criterion for the stability of discrete systems. Transforming (1) by the transformation \( \dot{x} = T_1 x \) where

\[
T_1 = \begin{bmatrix}
    1 & a_{1,n-1} & a_{2,n-1} & \cdots & a_{n-1,n-1} \\
    1 & a_{1,n-2} & a_{2,n-2} & \cdots & a_{n-2,n-2} \\
    \ddots & \ddots & \ddots & \ddots & \ddots \\
    1 & \ddots & \ddots & \ddots & \ddots \\
    \end{bmatrix}
\]

we get

\[
\dot{x}(k+1) = H_1 \dot{x}(k)
\]

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The Lyapunov method was published in [6] in 1965, which corresponds to different applications as estimation of the margin of stability, model reduction, and on recent instability results.

For $H_1$, the Lyapunov function $V = \dot{x}^T P_1 \dot{x}$ can be used

$$P_1 = \begin{bmatrix}
(1 - \Delta_n^2) & (1 - \Delta_n^2)(1 - \Delta_n^2) & \cdots & (1 - \Delta_n^2)(1 - \Delta_n^2)(1 - \Delta_n^2)
\end{bmatrix}$$

to give $\Delta V = -\dot{x}^T Q_1 \dot{x}$

$$Q_1 = \begin{bmatrix}
(1 - \Delta_n^2)^2 & 0 & \cdots & 0
\end{bmatrix}$$

which proves the Schur-Cohn criterion. For $H_2$:

$$P_2 = \begin{bmatrix}
\frac{1}{(1 - \Delta_n^2)} & \frac{1}{(1 - \Delta_n^2)(1 - \Delta_n^2)} & \cdots & \frac{1}{(1 - \Delta_n^2)(1 - \Delta_n^2)(1 - \Delta_n^2)}
\end{bmatrix}$$

and

$$Q_2 = \begin{bmatrix}
0 & \cdots & 0
\end{bmatrix}$$

It is to be noted that a projection of $H_1$ along the cross diagonal was found useful [10]. An extension to the case of complex coefficients was developed in [8]. In [9]-[11] the forms derived above were used for different applications as estimation of the margin of stability, model reduction, and recognition of the analogy between the equilibrium points in phase flows and the sinks of an incompressible fluid whose strength is characterized by the divergence of the velocity field $f$. The method of sinks is conceptually different from the Lyapunov-like procedures for estimation of stability regions near an equilibrium point. It is shown how to quantify the associated velocity fields. Several engineering examples have been solved which illustrate the usefulness of the technique.

### Planar Regions of Attraction

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Abstract—An effective method for estimation of the region of attraction of an equilibrium point in the plane is proposed in this note. This new method for stability analysis of the second-order nonlinear system $\dot{x} = f(x)$, $\chi(\lambda) = 0$, is based on the properties of phase flows and associated velocity fields. Several engineering examples have been solved which illustrate the usefulness of the technique.

I. INTRODUCTION

Stability analyses of nonlinear dynamic systems are usually conducted in order to determine the behavior of system trajectories in the neighborhood of singular solutions such as equilibrium points and periodic orbits. The global extent of this behavior is determined by estimation of stability regions in the state space. The method outlined below for estimation of regions of attraction in the plane is designated the "method of sinks" in recognition of the analogy between the equilibrium points in phase flows and the sinks of an incompressible fluid whose strength is characterized by the divergence of the velocity field $f$. The method of sinks is conceptually different from the Lyapunov-like procedures for estimation of stability regions (e.g., Davison and Cowan [2]). The method relies on Liouville's theorem which relates the div $f$ to the evolution of a volume of a state space domain subject to the phase flow, and on recent instability results obtained by Zhukov [7]. In addition to qualifying the behavior of trajectories near an equilibrium point, it is shown how to quantify the associated region of attraction. The applicability of the method of sinks to practical engineering problems is illustrated by examples.

III. COMPARISON TO THE RESULTS IN [11]

As shown above, the proof of the discrete stability criterion by means of Lyapunov method was published in [6] in 1965 which corresponds to the proof obtained by Parks [5] for the Routh-Hurwitz criterion. The matrix $A$ obtained in [11] is the same as $H_2$ in this note ($k_0 = \Delta_n$, $k_1 = \Delta_n - 1$, $\alpha_2 = \gamma_1 = l - k_1^2$, \ldots).

IV. CONCLUSIONS

In this note a survey of the results obtained by the author regarding the relation between the Schur-Cohn criterion and the Lyapunov method is given which shows that the proof of the stability of linear discrete systems by means of the Lyapunov method has been known for more than 16 years. The matrix obtained in [1] is the same as the matrix obtained before by the author. Moreover, a reference to different applications of this matrix form is given.

REFERENCES