Technical Notes and Correspondence

Generalized Lyapunov Function for Power Systems

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Abstract—A generalized Lyapunov function for power systems is obtained based on a previous work of the author. It is shown that it includes the different results in the literature and gives larger regions of asymptotic stability.

INTRODUCTION

In [1] a generalized Lyapunov function was obtained using the energy metric algorithm [2] for a two-machine system considering variable damping, transient saliency, flux decay, and governor action. Another generalized Lyapunov function was obtained for a multi-machine system with constant damping and not taking into consideration the other effects. It was shown in [1] that all the Lyapunov functions obtained so far for the above systems could be considered as special cases. They can be obtained if the free parameter in the generalized Lyapunov function is set equal to a certain constant. In a recent paper [3] Pai has obtained a Lyapunov function for power systems with uniform damping using a generalization of Popov's criterion which cannot be obtained from [1] by substituting a constant for the free parameter. However, following [1] another generalized Lyapunov function can be obtained which includes the one in [3] as a special case.

MATHEMATICAL MODEL

Following the notations in [1] a machine connected to an infinite busbar through a transmission line is described by the equations

\[ M\ddot{\delta} + D(\delta) \dot{\delta} = P_m(\delta) - P_e(E_{eq} \delta) \]

\[ E_{eq} = f(E_{eq}, \delta, \omega) \]

\[ P_m = f(P_m, \delta). \]

For a system with constant damping, no transient saliency, no flux decay, and neglecting governor action we get

\[ M\ddot{\delta} + ad\dot{\delta} = P_m - b\sin \delta, \quad P_m = b\sin \delta. \]

The system equations are

\[ \dot{x}_1 = x_2 \]

\[ M\dot{x}_2 = -ax_2 - b[\sin (x_2 + \delta_0) + \sin \delta_0]. \]

LYAPUNOV FUNCTION

Using the energy-metric algorithm we get the Lyapunov function

\[ V = \frac{1}{2} Mx^2 + b[\cos \delta_0 - \cos (x_2 + \delta_0) - x_2 \sin \delta_0]. \]

A generalization can be obtained by completing the quadratic terms

\[ V = ax_2^2 + bx_2x_3 + \frac{1}{2} Mx^2 + b[\cos \delta_0 - \cos (x_2 + \delta_0) - x_2 \sin \delta_0]. \]

The quadratic form is nonnegative if \( a \geq \beta^2/2M \).

In order to eliminate the term \( x_2x_3 \) in \( \dot{V} \) we set

\[ a = \frac{ab}{2M}, \quad 0 \leq \beta \leq \alpha. \]

\[ \dot{V} = -(a - \beta)x_2^2 - \frac{bg}{M} x_3[x_3 + \delta_0] - \sin \delta_0. \]  

The union of all the regions obtained for different values of the parameter \( \beta \) in the permissible range constitutes a region of asymptotic stability. For the more general system with variable damping, transient saliency, flux decay, and governor action see [1].

MULTIMACHINE SYSTEM

The equations of a simple multimachine system are given by

\[ M\ddot{x}_i + a_i\dot{x}_i + E_i g_i + \sum_{j \neq i} E_j R_{ij} \sin (\delta_i - \delta_j) = P_m \]

which can be reduced to the system equations

\[ M\ddot{x}_i = -a_i x_i - \sum_{j \neq i} b_{ij} [\sin (x_i - x_j + \delta_0 - \delta_i) - \sin (\delta_i - \delta_j)]. \]

Using the energy-metric algorithm we get

\[ V = \sum_{i=1}^{n} \frac{1}{2} Mx_i^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} b_{ij} [\cos (\delta_i - \delta_j) - \cos (x_i - x_j) \sin (\delta_i - \delta_j)]. \]

\[ \dot{V} = -\sum_{i=1}^{n} a_i x_i x_i. \]

A possible generalization can be obtained analog to (1) by completing the quadratic terms using only cross-products of the load-angle and the speed of the same machine.

\[ V = \beta \left[ \sum_{i=1}^{n} \frac{1}{2} a_i x_i^2 + \sum_{i=1}^{n} M_i x_i x_i \right] \]

\[ + \sum_{i=1}^{n} \frac{1}{2} M_i x_i x_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} b_{ij} [\cos (\delta_i - \delta_j) - \cos (x_i - x_j) \sin (\delta_i - \delta_j)]. \]

\[ \dot{V} = -\beta \sum_{i=1}^{n} \left( a_i - b M_i x_i \right) x_i. \]

(2)

\[ V \] is positive definite and \( \dot{V} \) is negative semi-definite or negative definite in a region around the origin, if

\[ 0 \leq \beta \leq \min \frac{a_i}{M_i} \]

for \( \delta_0 - \delta_i < \pi/2. \)

Special case

For \( a_i/M_i = \text{const} = \beta \) we get the Lyapunov function derived by Pai [3] for systems with uniform damping.

The Lyapunov functions (1),(2) and the generalized forms in [1] give results which include the previous results as special cases. The

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regions of asymptotic stability obtained by these generalized Lyapunov functions with free parameter are certainly larger than the regions obtained before. Combining the two generalized forms for multi-machine systems we get

\[ V = \alpha \left[ \sum_{i=1}^{n} (a_i \dot{x}_i + M \dot{x}_{i+n}) \right]^2 + \beta \left[ \sum_{i=1}^{n} \frac{a_i}{2} \dot{x}_i^2 + \sum_{i=1}^{n} M \dot{x}_{i+n} \right] + \sum_{i=1}^{n} \frac{1}{2} M \dot{x}_{i+n}^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \beta_{ij} \cos (\delta_{ij} - \delta_{ij}) - \cos (x_i - x_j + \delta_{ij} - \delta_{ij}) - (x_i - x_j) \sin (\delta_{ij} - \delta_{ij}) \]

\[ \dot{V} = - \sum_{i=1}^{n} (a_i - \beta M_i) \dot{x}_i^2 - \beta \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{x}_{ij} \sin (x_i - x_j + \delta_{ij} - \delta_{ij}) - \sin (\delta_{ij} - \delta_{ij}) \]

\[ 0 \leq \beta \leq \min \frac{\alpha}{M_i} \]

\[ \alpha \] has a negative lower limit, so that the quadratic form

\[ \alpha \left[ \sum_{i=1}^{n} (a_i \dot{x}_i + M \dot{x}_{i+n}) \right]^2 + \beta \left[ \sum_{i=1}^{n} \frac{a_i}{2} \dot{x}_i^2 + \sum_{i=1}^{n} M \dot{x}_{i+n} \right] + \sum_{i=1}^{n} \frac{1}{2} M \dot{x}_{i+n}^2 \]

is nonnegative.

**Conclusion**

Completing the quadratic terms in the Lyapunov function obtained by the energy metric algorithm for power systems gives a generalized Lyapunov function with one or more parameters. For certain values of the parameters the results in the literature can be obtained as special cases. The regions of asymptotic stability obtained by this generalized Lyapunov function are certainly larger than the regions obtained so far.

**REFERENCES**


**A Simplification of Jury's Tabular Form**

**ROBERT H. RAIBLE**

**Abstract**—A modification of Jury's procedure for the determination of the distribution of roots of polynomial \( F(z) \) with respect to the unit circle is described. Its advantages are simpler numerical manipulations and direct determination of root distribution in the so-called nonsingular cases, while previous methods for examination of singular cases are not subverted.

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**Modified Tabular Form**

Given the polynomial \( F(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \), the following table of calculated coefficients can be used to determine the distribution of the roots of \( F(z) \) with respect to the unit circle

\[
\begin{array}{cccccccc}
  a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 & k_1 & k_n \\
  b_0 & b_1 & b_2 & \cdots & b_{n-2} & b_{n-1} & k_1 & k_n \\
  c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} & k_1 & k_n \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
  f_0 & f_1 & f_2 & \cdots & f_{n-2} & f_{n-1} & 1 & k_n \\
\end{array}
\]

where

\[
b_0 = a_n/a_n, b_1 = b_{n-1}/b_n, k_1 = c_{n-2}/c_n, k_2 = 1/\xi_0 \\
b_0 = a_n - k_o a_n b_1 = a_{n-1} - k_o a_{n-1} b_{n-1} = a_2 - k_o a_{n-1} b_{n-1} = a_{n-2} - k_o a_{n-2} b_{n-2} = a_{n-3} - k_o a_{n-3} b_{n-3} = a_{n-4} - k_o a_{n-4} b_{n-4} = \cdots = a_1 - k_o a_1 b_1 = 1 - k_o a_1 b_1 = \xi_1 \\
\omega_0 = 1 - k_1^2 \\
\xi_0 = b_{n-1}/c_{n-2} \\
\delta_0 = a_{n-2}/a_{n-1} b_{n-2} c_{n-1} = a_{n-3}/a_{n-2} b_{n-3} c_{n-2} = a_{n-4}/a_{n-3} b_{n-4} c_{n-3} = a_{n-5}/a_{n-4} b_{n-5} c_{n-4} = \cdots = a_1/a_0 b_1 c_0 = 1/a_0 b_1 c_0 = \xi_1 \\
\omega_0 = b_{n-1}/c_{n-2} = \delta_0 \\
\]

**The Nonsingular Case**

If all of the calculated elements in the first column are nonzero (the nonsingular case) and if \( a_n \) is positive, the number of positive elements in the set \( (b_0, c_0, \cdots, \omega_0) \) indicates the number of roots of \( F(z) \) which are inside the unit circle and the number of negative elements then is equal to the number of roots outside the unit circle. The procedure indicated above was used by Aström [1] as a condition on stability but the present extension and interpretation, which appears to be a simplification of widely used methods, was not presented.

That the present interpretation of the above table is correct can be shown by comparison with the standard Jury procedure [2]. Using the designation \( h_i, b_i, c_i, \delta_i \) for the first column calculated elements of the Jury tabulation it can be shown by direct calculation that \( b_0 \) and every other element in the \( b \) row is \(-1/a_n\) times the corresponding element in the Jury table, first calculated row. \( c_0 \) and every other element in the \( c \) row is \(-1/a_n\) times the corresponding element in the second calculated row (Jury table).

Finally, comparing the algebraic signs of the parameters \( P_i, i = 1 - n_i \) of the Jury procedure to those of the set \( (b_0, c_0, \cdots, \omega_0) \) one finds the latter to be opposite in sign to the \( P_i \) and thus interpretation of \( (b_0, c_0, \cdots, \omega_0) \) is entirely similar, with the positive sign associated with roots of \( F(z) \) inside the unit circle.

**The Singular Case**

If, in the course of generating the modified tabular form described above, one encounters a value for \( k \) of plus or minus 1, the first element of the next row, and possibly other elements, will be zero, thus preventing further calculation. Resolution of the problem follows the