A NEW STABILITY ROBUSTNESS TEST FOR ADDITIVELY PERTURBED INTERCONNECTED SYSTEMS

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Abstract. This paper considers the stability robustness of additively perturbed systems using the small-µ methodology. The uncertainty \( \Delta(s) \) in the plant may perturb all the blocks of the monomial value \( P_0(s) \), and the norm of any block in \( \Delta(s) \) is bounded by \( \delta(j\omega) \) for some \( \mathcal{RH}_\infty \)-function \( \ell(s) \). The system is first transformed into an equivalent form in which the uncertainty has a block diagonal structure. The stability robustness is then characterized in terms of the structured singular value of a transfer function matrix \( H \). Some computational aspect of \( \mu(H) \) will be considered.

Keywords. interconnected systems, stability robustness, structured singular value

1 INTRODUCTION

The problem of robust stability has been studied by many authors, see e.g. Doyle and Stein (1981), Lehtomaki et al. (1981, 1984) and Postlethwaite et al. (1981). Referring to Fig. 1, we denote by \( P_0(s) \) the nominal plant and by \( \Delta(s) \) the unstructured additive perturbation. The controller \( C(s) \) stabilizes the nominal plant \( P_0(s) \). Then the perturbed system will remain stable if and only if

\[
\tilde{\sigma}[\Delta(j\omega)] < \sigma^{-1}[G(j\omega)]
\]

where \( \tilde{\sigma}[\cdot] \) is the maximum singular value, \( G(s) = C(I + P_0C)^{-1} \). Based on this inequality, the controller \( C(s) \) can be designed in such a way that the stability margin is maximized (see e.g. Glover 1986). However, this inequality does not take into account of the block structure of the uncertainty. Hence, as will be shown in section 2, it can be unnecessarily conservative. Further, it is impossible to characterize the stability robustness in terms of the blocks of the uncertainty. In Hung and Limbeer 1984 and Limebeer and Hung 1983, the Perren-Frobenius theory of nonnegative matrices was invoked to find some bounds of the stability robustness for interconnected systems containing a number of subsystems which are weakly coupled. However, the requirement that the subsystems be weakly coupled introduces a new restriction, the result is therefore also conservative. In Doyle (1983), the \( \mu \) function (structured singular value) was introduced for nonconservative robustness analysis and stabilizer design in case of structured (diagonal) plant uncertainties. In our previous work (Wu and Mansour 1991), we have shown that the uncertainty \( \Delta(s) \) can be transformed into block diagonal form. Hence a necessary and sufficient robust stability condition has been established in terms of the following structured singular value inequality:

\[
|\delta(j\omega)| < \mu^{-1}[H(j\omega)]
\]

where \( H(s) \) is a transfer function matrix which can be easily constructed from \( G(s) \). Since no weak coupling between the subsystems is presupposed, the new result can be applied to a wider range of interconnected systems.

However, the matrix \( H(s) \) has a much bigger dimension than \( G(s) \). In this paper, efforts will be devoted to find a matrix having the same dimension as \( G(s) \) but the same \( \mu \)-function as \( H(s) \).

This paper is organized as follows. Section 2 contains some preliminary results of robust stability and small \( \mu \) theory. In Section 3 we develop the main results of robust stability for interconnected systems. The main contribution of this paper and some problems, yet to be studied, are summarized in section 4.

On nomenclature: for any square complex matrix \( M \), we denote by \( \tilde{\sigma}[M] \) its maximum singular value, by \( M^* \) its complex conjugate transpose. Given any complex vector \( x \), \( x^* \) indicates its complex conjugate transpose and \( \|x\| \) its Euclidean norm. For any positive integer \( k \), we denote by \( 0_k \) the \( k \times k \) zero matrix, by \( I_k \) the \( k \times k \) identity matrix, by \( C_k \) the set of all \( k \) complex vectors and by \( C_k \) the set of all \( k \times k \) complex matrices. We call block structure of size \( m \) any \( m \)-tuple \( K = (k_1, k_2, \ldots, k_m) \). For any positive scalar \( \delta \) (possibly \( \infty \)), we denote by \( X_\delta \) the family of \( k \times k \) (\( k = \sum_{i=1}^{m} k_i \)) block diagonal matrices

\[
X_\delta = \{ \text{blockdiag}(\Delta_1, \Delta_2, \ldots, \Delta_n) : \Delta_i \in C^{k_i \times k_i}, \text{s.t.} \tilde{\sigma}[\Delta_i] \leq \delta \},
\]

and by \( Y \) the family of block unitary matrices

\[
Y = \{ \text{blockdiag}(V_1, V_2, \ldots, V_n) : V_i \text{ is a } k_i \times k_i \text{ unitary matrix} \}.
\]
2. PRELIMINARIES

Motivation

Consider the system in Fig. 1.

\[ P(s) = P_0(s) + \Delta(s) \]

where \( P(s) = P_0(s) + \Delta(s) \) is the plant having a nominal value at \( P_0(s) \). It is assumed that the uncertainty \( \Delta(s) \in \mathcal{RH}^\infty \). The controller \( C(s) \) stabilizes the nominal plant \( P_0(s) \). The following result is well known.

**Theorem 1** (Doyle and Stein 1981) The perturbed system remains stable if and only if

\[ \det (I + P_0 C + \epsilon \Delta C) \neq 0 \quad \forall \epsilon \in [0, 1] \quad \& \quad s = j\omega \]

where \( D \) is the Nyquist contour. For unstructured uncertainty \( \Delta(s) \), (1) is true if and only if

\[ \sigma[\Delta(j\omega)] < \sigma^{-1}[G(j\omega)] \quad \forall \omega \]

where \( G(s) = C(I + P_0 C)^{-1} \).

Now, let the system be composed of \( q \) interconnected subsystems. Then \( P_0(s) \) and \( \Delta(s) \) can be decomposed as follows:

\[ P_0(s) = \begin{pmatrix} P_{11}(s) & P_{12}(s) & \cdots & P_{1q}(s) \\ P_{21}(s) & P_{22}(s) & \cdots & P_{2q}(s) \\ \vdots & \vdots & \ddots & \vdots \\ P_{q1}(s) & P_{q2}(s) & \cdots & P_{qq}(s) \end{pmatrix} \]

\[ \Delta(s) = \begin{pmatrix} \Delta_{11}(s) & \Delta_{12}(s) & \cdots & \Delta_{1q}(s) \\ \Delta_{21}(s) & \Delta_{22}(s) & \cdots & \Delta_{2q}(s) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{q1}(s) & \Delta_{q2}(s) & \cdots & \Delta_{qq}(s) \end{pmatrix} \]

where \( P_{ij}(s) \) and \( \Delta_{ij}(s) \in (\mathcal{R}(j\omega))^{m \times m} \) for \( i = 1, 2, \ldots, q \) with \( q \ell = m \). \( \Delta_{ij}(s) \) are all in \( \mathcal{RH}^\infty \) and satisfy

\[ \sigma[\Delta_{ij}(j\omega)] \leq \delta(j\omega) \]

for some \( \mathcal{RH}^\infty \) function \( \delta(s) \) with minimum phase. Further, we partition \( G(s) \) as

\[ G(s) = \begin{pmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1q}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2q}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{q1}(s) & G_{q2}(s) & \cdots & G_{qq}(s) \end{pmatrix} \]

We shall show the conservatism of condition (2). For this purpose, let us suppose temporarily that the uncertainty \( \Delta(s) \) has identical blocks: \( \Delta_{ij}(s) = \Delta(s) \) for all \( i, j = 1, 2, \ldots, q \). Then

\[ \Delta(s) = J \hat{\Delta}(s) J^T \]

where

\[ J = [I_m \ 0 \ \cdots \ \ 0] \]

From (1) the perturbed system remains stable if and only if

\[ \det (I + P_0 G + \epsilon \hat{\Delta} G) \neq 0 \quad \forall \epsilon \in [0, 1] \quad \& \quad s = j\omega \]

Since

\[ \det(I + \epsilon \Delta G) = \det(I + \epsilon J \hat{\Delta} J^T G) \]

\[ = \det(I + \epsilon \hat{\Delta} J^T G J) \]

\[ = \det(I + \epsilon \hat{\Delta} \sum_{k,l=1}^q G_{kl}(j\omega)) \]

and \( \hat{\Delta}(s) \) is unstructured, we conclude that

\[ \det(I + \epsilon \hat{\Delta} G) \neq 0 \]

if and only if

\[ \sigma[\hat{\Delta}(j\omega)] < \sigma^{-1} \left[ \sum_{k,l=1}^q G_{kl}(j\omega) \right] \]

Now, we consider the system

\[ G(s) = \begin{pmatrix} G_S(s) & G_I(s) \\ -G_I^T(s) & -G_S(s) \end{pmatrix} \]

where \( G_S(s) \) and \( G_I(s) \) are SISO systems. It is readily verified that

\[ \sigma[G(j\omega)] = \max \{|G_S - G_I|, |G_S + G_I|\} \]

\[ \sigma[\Delta(j\omega)] = 2 |\hat{\Delta}(j\omega)| \]

From Eq. (2) the stability of the perturbed system can be guaranteed, if

\[ \sigma[\hat{\Delta}(j\omega)] < \frac{1}{2} \max \{|G_S - G_I|, |G_S + G_I|\} \]

However, since \( \sum_{i,j=1}^q G_{ij}(s) = 0 \), from the above analysis the perturbed system is stable for all \( \hat{\Delta}(s) \) such that \( \sigma[\hat{\Delta}(j\omega)] < \infty \). This example shows that the condition in Eq. (2) is really conservative.

2.2 The Structured Singular Value

In the following section, we will find some necessary and sufficient robust stability condition for interconnected systems. The results rely heavily on the concept of structured singular value. For this reason, its definition and some established results on this topics will be briefly presented in the following.

**Definition 1** (Doyle et al. 1982) The structured singular value \( \mu(H) \) of a \( k \times k \) matrix \( H \) with respect to block-structure \( \mathcal{K} \) is the positive number \( \mu \) having the following property that
\[
det(I + H\Delta_d) \neq 0 \quad \forall \Delta_d \in \mathcal{D}
if and only if 
\delta \mu(H) < 1.
\]
in other words,
\[
\mu(H) = \begin{cases} 
0 & \text{if } \det(I + H\Delta_d) \neq 0 \quad \forall \Delta_d \in \mathcal{D} \ni \\
\min \left\{ \sigma \left[ \Delta_d \right] : \exists \Delta_d \in \mathcal{D} \text{ s.t. } 
\det(I + H\Delta_d) = 0 \right\}^{-1} & \text{otherwise}
\end{cases}
\]

The computation of \( \mu(H) \) is equivalent to the solution of an eigenvalue optimization problem:

**Theorem 2** (Doyle 1982)

\[
\mu(H) = \max_{V \in \mathcal{V}} \rho(HV) = \max_{V \in \mathcal{V}} \rho(VH)
\]

where \( \rho(\cdot) \) is the spectral radius.

The following theorem is due to Fan and Tits (1986). It gives several equivalent expressions for the structured singular value.

**Theorem 3** The following two relations hold (and, in both, the maximum is achieved):

\[
\mu(H) = \max_{x \in \mathbb{C}^n} \left\{ \|Hx\| : \|S_i x\| < \|S_i Hx\|, \quad i = 1, 2, \ldots, n \right\}
\]

and

\[
\mu(H) = \max_{x \in \mathbb{C}^n} \left\{ \|Hx\| : \|S_i x\| < \|S_i Hx\|, \quad i = 1, 2, \ldots, n \right\}
\]

where \( S_i \) is the projection matrix

\[
S_i = \text{blockdiag}(0, \ldots, 0, I_i, 0, \ldots, 0).
\]

\[\Delta_d(s) = \text{blockdiag} \left( \Delta_1^t(s), \Delta_2^t(s), \ldots, \Delta_d^t(s) \right)\]

with

\[\Delta_1^t(s) = \text{blockdiag} \left( \Delta_1(s), \Delta_2(s), \ldots, \Delta_d(s) \right)\]

and

\[
M(s) = \begin{pmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{pmatrix}
\]

(8)

The transfer function matrix \( M(s) \) can be determined as follows. Denote

\[
F_l(M, \Delta_d) := M_{11} + M_{12} \Delta_d (I - M_{22} \Delta_d)^{-1} M_{21}.
\]

Then, being the transfer function matrix from \( v \) to \( y \), \( F_l(M, \Delta_d) \) must be equal to \( P_0(s) + \Delta(s) \), i.e.

\[
M_{11} + M_{12} \Delta_d (I - M_{22} \Delta_d)^{-1} M_{21} = P_0(s) + \Delta(s)
\]

This equality holds for

\[
M_{11}(s) = P_0(s)
\]

\[
M_{12}(s) = \text{blockdiag}(Y_{m_1}, Y_{m_2}, \ldots, Y_{m_q})
\]

\[
M_{21}(s) = (Z_{m_1} Z_{m_2} \ldots Z_{m_q})^t
\]

\[
M_{22}(s) = 0
\]

where

\[
Y_{m_i} = (I_{n_1} I_{n_2} \ldots I_{n_q})
\]

\[
Z_{m_i} = \text{blockdiag}(I_{n_1}, I_{n_2}, \ldots, I_{n_q})
\]

(9)

(10)

### 3 THE MAIN RESULTS

#### 3.1 Transform \( \Delta(s) \) into \( \Delta_d \)
In order to establish a robust stability test using the \( \mu \)-concept, we will first transform \( \Delta(s) \) into a block diagonal form.

**Fig. 2. A diagonally perturbed system**

The system in Fig. 1 can be represented by an equivalent form shown in Fig. 2, where

\[\det(I + H\Delta_d) \neq 0 \quad \forall \Delta_d \in \mathcal{D}\]

#### 3.2 A small-\( \mu \) test
From Theorem 1, the perturbed system remains stable if and only if

\[\det(I + c \Delta G) \neq 0 \quad \forall c \in [0, 1] \& s = j\omega\]

(11)

Substituting \( \Delta = M_{12} \Delta_d M_{21} \) into (11) and using the identity \( \det(I + XY) = \det(I + YX) \), we get

\[\det(I + c M_{12} \Delta_d M_{21} G) = \det(I + c \Delta_d H)\]

where

\[H(s) := M_{21} G(s) M_{12}\]

(12)

The following results are now evident.

**Theorem 4** (Wu and Mansour 1991) **The perturbed system with the controller \( C(s) \) remains stable if and only if**

\[\det(I + c \Delta_d H) \neq 0 \quad \forall c \in [0, 1] \& s = j\omega\]

**Or, equivalently,**

\[|\delta(j\omega)| < \mu^{-1}[H(j\omega)]\]

Theorem 4 established a necessary and sufficient robust stability condition for the interconnected system. However, in view of (9) and (10), the matrix \( H(s) \) has a much bigger dimension than the original system. In the sequel, it will be shown that there exists a transfer matrix having the same dimension as \( G(s) \) but the same \( \mu \)-function as \( H(s) \).
Theorem 5 Let \( U \) be the family of matrices
\[
\left\{ \begin{array}{c}
U_{11} & U_{12} & \cdots & U_{1q} \\
U_{21} & U_{22} & \cdots & U_{2q} \\
\vdots & \vdots & \ddots & \vdots \\
U_{q1} & U_{q2} & \cdots & U_{qq}
\end{array} \right\} : U_{ij} \text{ is unitary}
\]
where \( U_{ij} \in \mathbb{C}^{n \times n} \). Then
\[
\mu(H) = \max_{U \in U} \{ \mu(GU) \}
\]
proof. From Eq. (6) it is sufficient to prove that for every \( x \in \mathbb{C}^{q \cdot n^2} \)
\[
x = \left( \begin{array}{c}
z_1 \\
z_2 \\
z_3 \\
z_q 
\end{array} \right)
\]
where \( x_i \in \mathbb{C}^{n^2} \), satisfying
\[
\|S_i x_i\| H x_i = \|S_i H x_i\|, \quad i = 1, 2, \ldots, q\]
there exists a vector \( z \in \mathbb{C}^{q \cdot n^2} \)
\[
z = \left( \begin{array}{c}
z_1 \\
z_2 \\
z_3 \\
z_q 
\end{array} \right)
\]
where \( z_i \in \mathbb{C}^{n^2} \), such that
\[
\|S_i z_i\| \|GU z_i\| = \|S_i GU z_i\|, \quad i = 1, 2, \ldots, q
\]
for some \( U \in U \), where \( S_i \) is the projection matrix of dimension \((q \cdot n^2) \times (q \cdot n^2)\).
Routine matrix manipulations show
\[
y(x) := M_{12} x
\]
\[
y_i = \frac{x_i + x_{i+1} + x_{i+2} + \cdots + x_q}{x_{(q-1)i+1} + x_{(q-1)i+2} + \cdots + x_q}
\]
and
\[
\|S_1 x\| H x = \|S_1 H x\| = \|S_1 Gy(x)\|
\]
\[
\|S_2 x\| H x = \|S_2 H x\| = \|S_2 Gy(x)\|
\]
\[
\quad \vdots
\]
\[
\|S_q x\| H x = \|S_q H x\| = \|S_q Gy(x)\|
\]
\[
\|S_{q+1} x\| H x = \|S_{q+1} H x\| = \|S_{q+1} Gy(x)\|
\]
\[
\|S_{q+2} x\| H x = \|S_{q+2} H x\| = \|S_{q+2} Gy(x)\|
\]
\[
\quad \vdots
\]
\[
\|S_{2q} x\| H x = \|S_{2q} H x\| = \|S_{2q} Gy(x)\|
\]
\[
\quad \vdots
\]
Hence, for any \( l \in \{1, 2, \ldots, q\} \)
\[
\|x_l\| = \|x_{l+i}\| = \cdots = \|x_{(q-1)+i}\|\]
(15)
Denote by \( y_l \) the \( l \)th summand in (13)
\[
y_l = \left[ \begin{array}{c}
x_l \\
x_{l+1} \\
\vdots \\
x_{(q-1)+i}
\end{array} \right]
\]
for \( l = 1, 2, \ldots, q \), then from (15), we get
\[
y_l = \left( \begin{array}{c}
U_l \\
U_{q+1} \\
\vdots \\
U_{(q-1)+i}
\end{array} \right) z_l
\]
for some \( z_l \in \mathbb{C}^{n^2} \) and unitary matrices \( U_j \). Hence,
\[
y(x) = U z
\]
for some \( U \in U \) and \( z \) satisfies
\[
\|S_i z_i\| H x = \|S_i H x_i\|, \quad i = 1, 2, \ldots, q
\]
if and only if there exists a \( z \in \mathbb{C}^{n^2} \) and a \( U \in U \) such that
\[
\|S_i z_i\| \|GU z_i\| = \|S_i GU z_i\|, \quad i = 1, 2, \ldots, q
\]
Therefore,
\[
\mu(H) = \max_{\|x\| = 1} \{ \|H x\| : \|S_i x\| \|H x\| = \|S_i H x\|, \quad i = 1, 2, \ldots, q \}
\]
\[
= \max_{\|x\| = 1, U \in U} \{ \|GU z_i\| : \|S_i z_i\| \|GU z_i\| = \|S_i GU z_i\|, \quad i = 1, 2, \ldots, q \}
\]
\[
= \max_{U \in U} \{ \mu(GU) \}
\]
From theorem 2, we obtain
\[
\mu(H) = \max_{U \in U} \{ \|GU\| : \|S_i z_i\| \|GU z_i\| = \|S_i GU z_i\|, \quad i = 1, 2, \ldots, q \}
\]
Since for every \( U \in U \) and \( V \in \mathcal{V} \), \( UV \in U \), the following is obvious:

Corollary 1 Let \( U \) be defined in Theorem 5. Then
\[
\mu(H) = \max_{U \in U} \{ \mu(GU) \}
\]
Remark 1 It is interesting to compare corollary 1 with theorem 2. If \( \Delta(s) \) is already in block diagonal form, to compute \( \mu(G) \) we have to solve an eigenvalue optimization problem with respect to the set of (block) diagonal unitary matrices. If \( \Delta(s) \) is not of block diagonal form, we have then to solve the same problem but with respect to the set of matrices whose block entries are all unitary matrices.
3.3 Some bounds on $\mu(H)$

The purpose of this subsection is to deduce some lower bounds for $\mu(H)$. Hence necessary robust stability conditions will be established.

Corollary 2 (Wu and Mansour 1991)

\[
\mu[H(j\omega)] \geq \sigma \left[ \sum_{k=1}^{q} G_{k1}(j\omega) \right]
\]

Proof. We prove this corollary here using corollary 1. Also, let

\[
U_0 = \begin{bmatrix}
\bar{U} & \bar{U} & \cdots & \bar{U} \\
\bar{U} & \bar{U} & \cdots & \bar{U} \\
\vdots & \vdots & \ddots & \vdots \\
\bar{U} & \bar{U} & \cdots & \bar{U}
\end{bmatrix}
= I\bar{U} J^T
\]

It is clear that $U_0 \in \mathcal{U}$ for arbitrary unitary matrix $\bar{U}$. From corollary 1, we get

\[
\mu(H) = \max_{V \in \mathcal{U}} \rho(GU) \geq \max_{U_0} \rho(GU_0) = \max_{U_0} \rho(J^T GJU_0)
\]

It is readily verified that

\[
J^T G(j\omega) J = \sum_{k=1}^{q} G_{k1}(j\omega)
\]

Let $\sum_{k=1}^{q} G_{k1}(j\omega) = \tilde{H} \bar{U}$ be a polar decomposition. Then $\rho(\tilde{H}) = \sigma \left[ \sum_{i,j=1}^{q} G_{k1}(j\omega) \right]$. From

\[
\rho \left( \sum_{i,j=1}^{q} G_{k1}(j\omega) \bar{U} \right) \leq \sigma \left[ \sum_{i,j=1}^{q} G_{k1}(j\omega) \right]
\]

we see that

\[
\max_{\bar{U}} \rho(JGJ^T \bar{U}) = \sigma \left[ \sum_{i,j=1}^{q} G_{k1}(j\omega) \right]
\]

The optimal $\bar{U}$ is then $\bar{U}^*$. \hfill \Box

\textbf{Remark 2} Let $\sum_{i,j=1}^{q} G_{k1}(j\omega)$ have the SVD

\[
\sum_{i,j=1}^{q} G_{k1}(j\omega) = U \sigma \text{diag}[\sigma_1, \sigma_2, \ldots, \sigma_{\bar{m}}] V^*_c
\]

where $V_c$ and $U_c$ are unitary matrices, and

\[
\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\bar{m}} \geq 0
\]

Then the worst case destabilizing uncertainty is given by $\Delta(s) = J \Delta(s) J^T$, where $J$ is defined as before, $\Delta(s) \in R^H$ satisfying

\[
\Delta(j\omega) = V \sigma \text{diag}[-\sigma_1^{-1}, 0, \ldots, 0] V^*_c
\]

\textbf{Corollary 3} If the blocks of $G(s)$ are all SISO systems, then

\[
\mu[H(j\omega)] \geq \sum_{k=1}^{q} \left| \sum_{l=1}^{q} G_{k1}(j\omega) \right|
\]

and

\[
\mu[H(j\omega)] \geq \sum_{l=1}^{q} \left| \sum_{k=1}^{q} G_{k1}(j\omega) \right|
\]

Proof. We prove only the first inequality. Let

\[
U_0 = \begin{bmatrix}
e^{j\theta_1} & e^{j\theta_2} & \cdots & e^{j\theta_q} \\
e^{j\theta_1} & e^{j\theta_2} & \cdots & e^{j\theta_q} \\
\vdots & \vdots & \ddots & \vdots \\
e^{j\theta_1} & e^{j\theta_2} & \cdots & e^{j\theta_q}
\end{bmatrix}
\]

where

\[
\bar{\mathbf{q}} = [\theta_1, \theta_2, \ldots, \theta_q]
\]

is the parameter vector to be determined. It is clear that $U_0 \in \mathcal{U}$ and

\[
GU_0 = gV^T
\]

where

\[
g = \left[ \sum_{l=1}^{q} G_{1l}, \sum_{l=1}^{q} G_{2l}, \ldots, \sum_{l=1}^{q} G_{ql} \right]^T
\]

\[
v = \left[ e^{j\theta_1}, e^{j\theta_2}, \ldots, e^{j\theta_q} \right]^T
\]

From corollary 1

\[
\mu(H) = \max_{V \in \mathcal{U}} \rho(GU) \geq \max_{\bar{\mathbf{q}}} \{ \rho(GU_0) \}
\]

\[
= \max_{\bar{\mathbf{q}}} \{ v^T g \}
\]

\[
= \max_{\bar{\mathbf{q}}} \left[ \sum_{k=1}^{q} \left| \sum_{l=1}^{q} G_{k1}(j\omega) \right| e^{j\theta_k} \right]
\]

Let

\[
\sum_{l=1}^{q} G_{k1}(j\omega) = \sum_{l=1}^{q} G_{k1}(j\omega) e^{j\alpha_k}
\]

then the optimal $\bar{\mathbf{q}}$ is given by

\[
\bar{\theta}_0 = [-\alpha_1 - \alpha_2 - \ldots - \alpha_q]
\]

and

\[
\max_{U_0} \{ \rho(GU_0) \} = \sum_{k=1}^{q} \left| \sum_{l=1}^{q} G_{k1}(j\omega) \right|
\]

Since $U_0 \in \mathcal{U}$, there holds

\[
\max_{U \in \mathcal{U}} \{ \rho(GU) \} \geq \rho(GU_0)
\]

and the result follows. \hfill \Box

\textbf{Remark 3} If for some $s = j\omega$, $|\mathbf{d}(j\omega)| = \left( \sum_{k=1}^{q} \sum_{l=1}^{q} G_{k1}(j\omega) \right)^{-1}$, the uncertainty
$\Delta = -|\delta(j\omega)| \begin{bmatrix} e^{-j\omega_1} & e^{-j\omega_2} & \cdots & e^{-j\omega_q} \\ e^{j\omega_1} & e^{j\omega_2} & \cdots & e^{j\omega_q} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-j\omega_1} & e^{-j\omega_2} & \cdots & e^{-j\omega_q} \end{bmatrix}$

will destabilize the closed-loop system. Indeed, for $s = j\omega$, $\Delta(s)$ can be represented as

$\Delta(j\omega) = -|\delta(j\omega)| v_0 v_0^T$

where

$u_0 = [1 \ 1 \ \ldots \ 1]^T$

$v_0 = [e^{-j\omega_1} \ e^{-j\omega_2} \ \ldots \ e^{-j\omega_q}]^T$

Then

$\det(I + \Delta G) = 1 - |\delta(j\omega)| v_0 G u_0$

$= 1 - |\delta(j\omega)| v_0^T G$

$= 1 - |\delta(j\omega)| |\delta(j\omega)|^{-1} = 0$

The perturbed system will have a pole at $s = j\omega$.

Remark 4 For multivariable $G_i(s)$, the result in corollary 3 can be generalized as follows. Let

$\sum_{k=1}^q G_i(j\omega) = U_i \cdot H_i$

where $U_i$ is unitary, $H_i \geq 0$. Then

$\nu[H(j\omega)] \geq \sigma \left[ \sum_{i=1}^q H_i \right]$

Let $\bar{H} := \sum_{i=1}^q H_i$ have the singular value decomposition

$\bar{H} = \bar{U} \Sigma \bar{U}^*$

where $\bar{U}$ is unitary and

$\Sigma = \text{diag} [\sigma_1, \sigma_2, \ldots, \sigma_m]$

with

$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0$

Then, a destabilizing uncertainty can be constructed as

$\Delta(j\omega) = \hat{\Delta} \begin{bmatrix} U_1^* & U_2^* & \cdots & U_q^* \\ U_1^* & U_2^* & \cdots & U_q^* \\ \vdots & \vdots & \ddots & \vdots \\ U_1^* & U_2^* & \cdots & U_q^* \end{bmatrix}$

with

$\hat{\Delta} = \bar{U} \text{diag} [-\sigma_1^{-1}, 0, \ldots, 0] \bar{U}^*$

4 CONCLUSION

In this paper it was shown that the computation of $\nu[H(j\omega)]$ is equivalent to an eigenvalue optimization problem. Unlike the case where $\Delta(s)$ is already in block diagonal form, the optimization has to be performed with respect to the set of matrices whose block entries are all unitary matrices. We have also shown that in some special cases $\nu[H(j\omega)]$ has an analytical solution. The corresponding worst case perturbation has been characterized.

REFERENCES


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A NEW STABILITY ROBUSTNESS TEST FOR ADDITIVELY PERTURBED INTERCONNECTED SYSTEMS

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Abstract. This paper considers the stability robustness of additively perturbed systems using the small-$\mu$ methodology. The uncertainty $\Delta(s)$ in the plant may perturb all the blocks of the monomial value $P_0(s)$, and the norm of any block in $\Delta(s)$ is bounded by $\sigma(\Delta(s)) \leq |\delta(i\omega)|$ for some $\mathcal{RH}_\infty$-function $\delta(s)$. The system is first transformed into an equivalent form in which the uncertainty has a block diagonal structure. The stability robustness is then characterized in terms of the structured singular value of a transfer function matrix $H$. Some computational aspect of $\mu(H)$ will be considered.

Keywords. Interconnected systems, stability robustness, structured singular value

1 INTRODUCTION

The problem of robust stability has been studied by many authors, see e.g. Doyle and Stein (1981), Lehtomaki et al. (1981, 1984) and Postlethwaite et al. (1981). Referring to Fig. 1, we denote by $P_0(s)$ the nominal plant and by $\Delta(s)$ the unstructured additive perturbation. The controller $C(s)$ stabilizes the nominal plant $P_0(s)$. Then the perturbed system will remain stable if and only if

$$\sigma(\Delta(i\omega)) < \sigma^{-1}[G(i\omega)]$$

where $\sigma(.)$ is the maximum singular value, $G(s) = C(I + P_0C)^{-1}$. Based on this inequality, the controller $C(s)$ can be designed in such a way that the stability margin is maximized (see e.g. Glover 1986). However, this inequality does not take into account of the block structure of the uncertainty. Hence, as will be shown in section 2, it can be unnecessarily conservative. Further, it is impossible to characterize the stability robustness in terms of the blocks of the uncertainty. In Hung and Limebeer 1984 and Limebeer and Hung 1983, the Perron-Frobenius theory of nonnegative matrices was invoked to find some bounds of the stability robustness for interconnected systems containing a number of subsystems which are weakly coupled. However, the requirement that the subsystems be weakly coupled introduces a new restriction, the result is therefore also conservative. In Doyle (1983), the $\mu$ function (structured singular value) was introduced for nonconservative robustness analysis and stabilizer design in case of structured (diagonal) plant uncertainties. In our previous work (Wu and Mansour 1991), we have shown that the uncertainty $\Delta(s)$ can be transformed into block diagonal form. Hence a necessary and sufficient robust stability condition has been established in terms of the following structured singular value inequality:

$$|\delta(i\omega)| < \mu^{-1}[H(i\omega)]$$

where $H(s)$ is a transfer function matrix which can be easily constructed from $G(s)$. Since no weak coupling between the subsystems is presupposed, the new result can be applied to a wider range of interconnected systems.

However, the matrix $H(s)$ has a much bigger dimension than $G(s)$. In this paper, efforts will be devoted to find a matrix having the same dimension as $G(s)$ but the same $\mu$-function as $H(s)$. This paper is organized as follows. Section 2 contains some preliminary results of robust stability and small $\mu$ theory. In Section 3 we develop the main results of robust stability for interconnected systems. The main contribution of this paper and some problems, yet to be studied, are summarized in section 4.

On nomenclature: for any square complex matrix $M$, we denote by $\sigma(M)$ its maximum singular value, by $M^*$ its complex conjugate transpose. Given any complex vector $x$, $x^*$ indicates its complex conjugate transpose and $\|x\|$ its Euclidean norm. For any positive integer $k$, we denote by $I_k$ the $k \times k$ zero matrix, by $I_k$ the $k \times k$ identity matrix, by $C_k$ the set of all $k$ complex vectors and by $C_k^d$ the set of all $k \times k$ complex matrices. We call block structure of size $m$ any $m$-tuple $K = (k_1, k_2, \ldots, k_m)$. For any positive scalar $\delta$ (possibly $\infty$), we denote by $X_\delta$ the family of $k \times k$ ($k = \sum_{i=1}^m k_i$) block diagonal matrices

$$X_{\delta} = \{\text{blockdiag}(\Delta_1, \Delta_2, \ldots, \Delta_n) : \Delta_i \in C_{k_i \times k_i}^d \text{ s.t. } \sigma(\Delta_i) \leq \delta\}$$

and by $\mathcal{V}$ the family of block unitary matrices

$$\mathcal{V} = \{\text{blockdiag}(V_1, V_2, \ldots, V_n) : V_i \text{ is a } k_i \times k_i \text{ unitary matrix}\}.$$
2 PRELIMINARIES

2.1 Motivation

We consider the system in Fig. 1.

![System Diagram]

Fig. 1. The system of concern

Here $P(s) = P_0(s) + \Delta(s)$ is the plant having a nominal value at $P_0(s)$. It is assumed that the uncertainty $\Delta(s) \in \mathcal{RH}_\infty$. The controller $G(s)$ stabilizes the nominal plant $P_0(s)$. The following result is well known.

Theorem 1 (Doyle and Stein 1981) The perturbed system remains stable if and only if

$$\det(I + P_0C + \epsilon\Delta C) \neq 0 \quad \forall \epsilon \in [0, 1] \text{ & } s \in D(1)$$

where $D$ is the Nyquist contour. For unstructured uncertainty $\Delta(s)$, (1) is true if and only if

$$\sigma(\Delta(j\omega)) < \sigma^{-1}[G(j\omega)] \quad \forall \omega$$

(2)

where $G(s) = C(I + P_0C)^{-1}$.

Now, let the system be composed of $q$ interconnected subsystems. Then $P_0(s)$ and $\Delta(s)$ can be decomposed as follows:

$$P_0(s) = \begin{pmatrix}
P_{11}(s) & P_{12}(s) & \ldots & P_{1q}(s) \\
P_{21}(s) & P_{22}(s) & \ldots & P_{2q}(s) \\
\vdots & \vdots & \ddots & \vdots \\
P_{q1}(s) & P_{q2}(s) & \ldots & P_{qq}(s)
\end{pmatrix}$$

(3)

and

$$\Delta(s) = \begin{pmatrix}
\Delta_{11}(s) & \Delta_{12}(s) & \ldots & \Delta_{1q}(s) \\
\Delta_{21}(s) & \Delta_{22}(s) & \ldots & \Delta_{2q}(s) \\
\vdots & \vdots & \ddots & \vdots \\
\Delta_{q1}(s) & \Delta_{q2}(s) & \ldots & \Delta_{qq}(s)
\end{pmatrix}$$

(4)

Here $P_{ii}(s)$ and $\Delta_{ii}(s) \in (\mathcal{RH}_\infty)^{m \times m}$ for $i = 1, 2, \ldots, q$ with $m \in \mathbb{N}$. $\Delta_{ij}(s)$ are all in $\mathcal{RH}_\infty$ and satisfy

$$\sigma(\Delta_{ij}(j\omega)) \leq |\delta(j\omega)|$$

for some $\mathcal{RH}_\infty$ function $\delta(s)$ with minimum phase. Further, we partition $G(s)$ as

$$G(s) = \begin{pmatrix}
G_{11}(s) & G_{12}(s) & \ldots & G_{1q}(s) \\
G_{21}(s) & G_{22}(s) & \ldots & G_{2q}(s) \\
\vdots & \vdots & \ddots & \vdots \\
G_{q1}(s) & G_{q2}(s) & \ldots & G_{qq}(s)
\end{pmatrix}$$

(5)

We shall show the conservatism of condition (2). For this purpose, let us suppose temporarily that the uncertainty $\Delta(s)$ has identical blocks: $\Delta_{ij}(s) = \hat{\Delta}(s)$ for all $i, j = 1, 2, \ldots, q$. Then

$$\Delta(s) = J\hat{\Delta}(s)J^T$$

where

$$J = \begin{pmatrix}
I_m & I_m & \ldots & I_m
\end{pmatrix}^T$$

From (1) the perturbed system remains stable if and only if

$$\det(I + \epsilon\hat{\Delta}G) \neq 0 \quad \forall \epsilon \in [0, 1] \text{ & } s = j\omega$$

Since

$$\det(I + \epsilon\Delta G) = \det(I + \epsilon J\hat{\Delta}J^TG)$$

$$= \det(I + \epsilon \hat{\Delta}J^TGJ)$$

$$= \det(I + \epsilon \sum_{k,l=1}^{q} G_{kl}(j\omega))$$

and $\hat{\Delta}(s)$ is unstructured, we conclude that

$$\det(I + \epsilon\hat{\Delta}G) \neq 0$$

if and only if

$$\sigma(\hat{\Delta}(j\omega)) < \sigma^{-1}\left(\sum_{k,l=1}^{q} G_{kl}(j\omega)\right)$$

Now, we consider the system

$$G(s) = \begin{pmatrix}
G_S(s) & G_I(s) \\
-G_I(s) & -G_S(s)
\end{pmatrix}$$

where $G_S(s)$ and $G_I(s)$ are SISO systems. It is readily verified that

$$\sigma[G(j\omega)] = \max\{|G_S - G_I|, |G_S + G_I|\}$$

$$\sigma[\Delta(j\omega)] = 2|\hat{\Delta}(j\omega)|$$

From Eq. (2) the stability of the perturbed system can be guaranteed, if

$$\sigma[\hat{\Delta}(j\omega)] < \frac{1}{2\max\{|G_S - G_I|, |G_S + G_I|\}}$$

However, since $\sum_{i,j=1}^{q} G_{ij}(s) = 0$, from the above analysis the perturbed system is stable for all $\hat{\Delta}(s)$ such that $\sigma[\hat{\Delta}(j\omega)] < \infty$. This example shows that the condition in Eq. (2) is really conservative.

2.2 The Structured Singular Value

In the following section, we will find some necessary and sufficient robust stability condition for interconnected systems. The results rely heavily on the concept of structured singular value. For this reason, its definition and some established results on this topic will be briefly presented in the following.

Definition 1 (Doyle et al. 1982) The structured singular value $\mu(H)$ of a $k \times k$ matrix $H$ with respect to block-structure $K$ is the positive number $\mu$ having the following property that
\[ \det(I + H \Delta_d) \neq 0 \quad \forall \Delta_d \in \mathcal{X}_d \]
if and only if
\[ \delta \mu(H) < 1. \]
in other words,
\[ \mu(H) = \begin{cases} 0 & \text{if } \det(I + H \Delta_d) \neq 0 \quad \forall \Delta_d \in \mathcal{X}_d \\ \min \{ \sigma(\Delta_d) : \exists \Delta_d \in \mathcal{X}_d \text{ s.t. } \det(I + H \Delta_d) = 0 \} & \text{otherwise} \end{cases} \]

The computation of \( \mu(H) \) is equivalent to the solution of an eigenvalue optimization problem:

Theorem 2 (Doyle 1982)
\[ \mu(H) = \max_{V \in \mathbb{C}^n} \rho(HV) = \max_{V \in \mathbb{C}^n} \rho(VH) \]
where \( \rho(\cdot) \) is the spectral radius.

The following theorem is due to Fan and Tits (1986). It gives several equivalent expressions for the structurally singular value.

Theorem 3 The following two relations hold (and, in both, the maximum is achieved):
\[ \mu(H) = \max_{x \in \mathbb{C}^n} \{ \|Hx\| : \|S_ix\| \leq \|S_iHx\|, \quad i = 1, 2, \ldots, n \} \]
and
\[ \mu(H) = \max_{x \in \mathbb{C}^n} \{ \|Hx\| : \|S_ix\| \leq \|S_iHx\|, \quad i = 1, 2, \ldots, n \} \]
where \( S_i \) is the projection matrix
\[ S_i = \text{blockdiag}(0, \ldots, 0, I_{k_i}, 0, \ldots, 0) \].

3 THE MAIN RESULTS

3.1 Transform \( \Delta(s) \) into \( \Delta_d \)
In order to establish a robust stability test using the \( \mu \)-concept, we will first transform \( \Delta(s) \) into a block diagonal form.

\[ \Delta_d(s) = \text{blockdiag}(\Delta^1(s), \Delta^2(s), \ldots, \Delta^q(s)) \]
with
\[ \Delta^i(s) = \text{blockdiag}(\Delta_{1i}(s), \Delta_{2i}(s), \ldots, \Delta_{10i}(s)) \]
and
\[ M(s) = \begin{pmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{pmatrix} \] (8)
The transfer function matrix \( M(s) \) can be determined as follows. Denote
\[ F_1(M, \Delta_d) := M_{11} + M_{12}\Delta_d(I - M_{22}\Delta_d)^{-1}M_{21}. \]
Then, the transfer function matrix from \( v \) to \( y \), \( F_1(M, \Delta_d) \) must be equal to \( P_0(s) + \Delta(s) \), i.e.
\[ M_{11} + M_{12}\Delta_d(I - M_{22}\Delta_d)^{-1}M_{21} = P_0(s) + \Delta(s) \]
This equality holds for
\[ M_{11}(s) = P_0(s) \]
\[ M_{12}(s) = \text{blockdiag}(Y_m, Y_m, \ldots, Y_m) \]
\[ M_{21}(s) = (Z_m Z_m \ldots Z_m)^{-1} \]
\[ M_{22}(s) = 0 \]
where
\[ Y_m = (I_{n_1}, I_{n_2}, \ldots, I_{n_q}) \]
\[ Z_m = \text{blockdiag}(I_{n_1}, I_{n_2}, \ldots, I_{n_q}) \] (9)

3.2 A small-\( \mu \) test
From Theorem 1, the perturbed system remains stable if and only if
\[ \det(I + \epsilon \Delta G) \neq 0 \quad \forall \epsilon \in [0, 1] \& s = j\omega \] (11)
Substituting \( \Delta = M_{12}\Delta_dM_{21} \) into (11) and using the identity \( \det(I + XY) = \det(I + YX) \), we get
\[ \det(I + \epsilon M_{12}\Delta_dM_{21}G) = \det(I + \epsilon \Delta_dH) \]
where
\[ H(s) := M_{21}G(s)M_{12}. \] (12)
The following results is now evident.

Theorem 4 (Wu and Mansour 1991) The perturbed system with the controller \( C(s) \) remains stable if and only if
\[ \det(I + \epsilon \Delta_dH) \neq 0 \quad \forall \epsilon \in [0, 1] \& s = j\omega \]
Or, equivalently,
\[ |\delta(j\omega)| < \mu^{-1}(|H(j\omega)|) \]

Theorem 4 established a necessary and sufficient robust stability condition for the interconnected system. However, in view of (9) and (10), the matrix \( H(s) \) has a much bigger dimension than the original system. In the sequel, it will be shown that there exists a transfer matrix having the same dimension as \( G(s) \) but the same \( \mu \)-function as \( H(s) \).
Theorem 5 Let $U$ be the family of matrices
\[
\begin{pmatrix}
U_{11} & U_{12} & \ldots & U_{1q} \\
U_{21} & U_{22} & \ldots & U_{2q} \\
\vdots & \vdots & \ddots & \vdots \\
U_{q1} & U_{q2} & \ldots & U_{qq}
\end{pmatrix}
: U_{ij} \text{ is unitary}
\]
where $U_{ij} \in \mathbb{C}^{n \times n}$. Then
\[
\mu(H) = \max_{U \in U} \mu(GU)
\]
proof. From Eq. (6) it is sufficient to prove that for every $x \in \mathbb{C}^{n_2}$
\[
x = \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_{q^2}
\end{pmatrix}
\]
where $x_i \in \mathbb{C}^{n_i}$, satisfying
\[
\|S_i x\| \leq \|H x\| = \|S_i H x\|, \quad i = 1, 2, \ldots, q^2
\]
there exists a vector $z \in \mathbb{C}^{n_2}$
\[
z = \begin{pmatrix}
z_1 \\
z_2 \\
\vdots \\
z_{q^2}
\end{pmatrix}
\]
where $z_i \in \mathbb{C}^{n_i}$, such that
\[
\|\tilde{S}_i z\| \leq \|GU z\| = \|\tilde{S}_i G U z\|, \quad i = 1, 2, \ldots, q
\]
for some $U \in U$, where $\tilde{S}_i$ is the projection matrix of dimension $(q_1 \tilde{m}) \times (q_1 \tilde{m})$.
Routine matrix manipulation show
\[
\tilde{y}(x) := M_{12} x
\]
\[
= \begin{pmatrix}
x_1 + x_2 + \ldots + x_q \\
x_{q+1} + x_{q+2} + \ldots + x_{2q} \\
\vdots \\
x_{(q-1)q+1} + x_{(q-1)q+2} + \ldots + x_{q^2}
\end{pmatrix}
\]
and
\[
\|S_1 x\| \leq \|H x\| = \|S_1 H x\| = \|S_1 G \tilde{y}(x)\|
\]
\[
\|S_2 x\| \leq \|H x\| = \|S_2 H x\| = \|S_2 G \tilde{y}(x)\|
\]
\[
\vdots
\]
\[
\|S_q x\| \leq \|H x\| = \|S_q H x\| = \|S_q G \tilde{y}(x)\|
\]
\[
\|S_{q+1} x\| \leq \|H x\| = \|S_{q+1} H x\| = \|S_{q+1} G \tilde{y}(x)\|
\]
\[
\|S_{q+2} x\| \leq \|H x\| = \|S_{q+2} H x\| = \|S_{q+2} G \tilde{y}(x)\|
\]
\[
\vdots
\]
\[
\|S_{q^2} x\| \leq \|H x\| = \|S_{q^2} H x\| = \|S_{q^2} G \tilde{y}(x)\|
\]
Hence, for any $l \in \{1, 2, \ldots, q\}$
\[
\|x_l\| = \|x_{q+l}\| = \cdots = \|x_{(q-1)q+l}\|
\]
Denote by $y_l$ the $l$th summand in (13)
\[
y_l = \begin{pmatrix}
x_1 \\
x_{q+1} \\
\vdots \\
x_{(q-1)q+1}
\end{pmatrix}
\]
for $l = 1, 2, \ldots, q$, then from (15), we get
\[
y_l = \begin{pmatrix}
U_l \\
U_{q+l} \\
\vdots \\
U_{(q-1)q+l}
\end{pmatrix}
z_l
\]
for some $z_l \in \mathbb{C}^{n_l}$ and unitary matrices $U_l$. Hence,
\[
\tilde{y}(x) = U z
\]
for some $U \in U$ and $x$ satisfies
\[
\|S_i x\| \leq \|H x\| = \|S_i H x\|, \quad i = 1, 2, \ldots, q^2
\]
if and only if there exists a $z \in \mathbb{C}^n$ and a $U \in U$ such that
\[
\|\tilde{S}_i z\| \leq \|GU z\| = \|\tilde{S}_i G U z\|, \quad i = 1, 2, \ldots, q
\]
Therefore,
\[
\mu(H) = \max_{x \in \mathbb{C}^{n_2}} \{\|H x\| : \|S_i x\| \leq \|H x\|, \quad i = 1, 2, \ldots, q^2\}
\]
\[
= \max_{x \in \mathbb{C}^{n_2}, U \in U} \{\|GU x\| : \|\tilde{S}_i z\| \leq \|GU z\|, \quad i = 1, 2, \ldots, q\}
\]
\[
= \max_{U \in U} \{\mu(GU)\}
\]
From theorem 2, we obtain
\[
\mu(H) = \max_{V \in V} \max_{U \in U} \{\rho(GU V)\}
\]
Since for every $U \in U$ and $V \in V$, $UV \in U$, the following is obvious:

Corollary 1 Let $U$ be defined in theorem 5. Then
\[
\mu(H) = \max_{U \in U} \mu(GU)
\]

Remark 1 It is interesting to compare corollary 1 with theorem 2. If $\Delta(x)$ is already in block diagonal form, to compute $\mu(G)$ we have to solve an eigenvalue optimization problem with respect to the set of (block) diagonal unitary matrices. If $\Delta(x)$ is not of block diagonal form, we have then to solve the same problem but with respect to the set of matrices whose block entries are all unitary matrices.
3.3 Some bounds on $\mu(H)$

The purpose of this subsection is to deduce some lower bounds for $\mu(H)$. Hence necessary robust stability conditions will be established.

Corollary 2 (Wu and Mansour 1991)

$$\mu[H(j\omega)] \geq \delta \left[ \sum_{k=1}^{q} G_{kl}(j\omega) \right]$$

**Proof.** We prove this corollary here using corollary 1. Also let

$$U_0 = \begin{bmatrix}
\hat{U} & \hat{U} & \cdots & \hat{U} \\
\hat{U} & \hat{U} & \cdots & \hat{U} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{U} & \hat{U} & \cdots & \hat{U}
\end{bmatrix}
= J\hat{U}J^T$$

It is clear that $U_0 \in \mathcal{U}$ for arbitrary unitary matrix $\hat{U}$. From corollary 1, we get

$$\mu(H) = \max_{\tilde{U} \in \mathcal{U}} \rho(GU) = \max_{\tilde{U} \in \mathcal{U}} \rho(GU_0) = \max_{\tilde{U} \in \mathcal{U}} \rho(J^T GJ\hat{U})$$

It is readily verified that

$$J^T G(j\omega)J = \sum_{k=1}^{q} G_{kl}(j\omega)$$

Let $\sum_{k=1}^{q} G_{kl}(j\omega) = \hat{R}\hat{U}$ be a polar decomposition. Then $\rho(\hat{R}) = \delta \left[ \sum_{i,j=1}^{q} G_{kl}(j\omega) \right]$. From

$$\rho \left( \sum_{i,j=1}^{q} G_{kl}(j\omega)\hat{U} \right) \leq \delta \left[ \sum_{i,j=1}^{q} G_{kl}(j\omega) \right]$$

we see that

$$\max_{\tilde{U}} \rho(JGJ^T\hat{U}) = \delta \left[ \sum_{i,j=1}^{q} G_{kl}(j\omega) \right]$$

The optimal $\hat{U}$ is then $\hat{U}^*$. □

Remark 2 Let $\sum_{k,l=1}^{q} G_{kl}(j\omega)$ have the SVD

$$\sum_{k,l=1}^{q} G_{kl}(j\omega) = U_s\text{diag}[^{\sigma_1}_1, \sigma_2, \ldots, \sigma_m] V_s^*$$

where $V_s$ and $U_s$ are unitary matrices, and

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_m \geq 0$$

Then the worst case destabilizing uncertainty is given by $\Delta(s) = J\Delta^*(s)J^T$, where $J$ is defined as before, $\Delta(s) \in \mathcal{RH}$ satisfying

$$\Delta(j\omega) = V_s\text{diag}[-^{\sigma_1}_1, 0, \ldots, 0] U_s^*$$

Corollary 3 If the blocks of $G(s)$ are all SISO systems, then

$$\mu[H(j\omega)] \geq \sum_{k=1}^{q} \sum_{l=1}^{q} G_{kl}(j\omega)$$

and

$$\mu[H(j\omega)] \geq \sum_{k=1}^{q} \sum_{l=1}^{q} G_{kl}(j\omega)$$

**Proof.** We prove only the first inequality. Let

$$U_0 = \begin{bmatrix}
e^{j\theta_1} & e^{j\theta_2} & \cdots & e^{j\theta_q} \\
e^{j\theta_1} & e^{j\theta_2} & \cdots & e^{j\theta_q} \\
\vdots & \vdots & \ddots & \vdots \\
e^{j\theta_1} & e^{j\theta_2} & \cdots & e^{j\theta_q}
\end{bmatrix},$$

where

$$\tilde{\theta} = [\theta_1, \theta_2, \ldots, \theta_q]$$

is the parameter vector to be determined. It is clear that $U_0 \in \mathcal{U}$ and

$$GU_0 = g\nu^T$$

where

$$g = \left[ \sum_{i=1}^{q} G_{11}, \sum_{i=1}^{q} G_{21}, \ldots, \sum_{i=1}^{q} G_{q1} \right]^T$$

$$\nu = \left[ e^{j\theta_1}, e^{j\theta_2}, \ldots, e^{j\theta_q} \right]^T$$

From corollary 1

$$\mu(H) = \max_{\tilde{U} \in \mathcal{U}} \rho(GU) \geq \max_{\tilde{U} \in \mathcal{U}} \{ \rho(GU_0) \}$$

$$= \max_{\theta} \{ \nu^T g \}$$

$$= \max_{\theta} \sum_{k=1}^{q} \left[ \sum_{l=1}^{q} G_{kl}(j\omega) \right] e^{j\theta_k}$$

Let

$$\sum_{k=1}^{q} \sum_{l=1}^{q} G_{kl}(j\omega) = \left[ \sum_{k=1}^{q} G_{kl}(j\omega) \right] e^{j\theta_k}$$

then the optimal $\theta$ is given by

$$\theta = [-\sigma_1, -\sigma_2, \ldots, -\sigma_q]$$

and

$$\max_{\theta} \{ \rho(GU_0) \} = \left[ \sum_{k=1}^{q} \sum_{l=1}^{q} G_{kl}(j\omega) \right]$$

Since $U_0 \in \mathcal{U}$, there holds

$$\max_{\tilde{U} \in \mathcal{U}} \{ \rho(GU) \} \geq \rho(GU_0)$$

and the result follows. □

Remark 3 If for some $s = j\omega$, $|\delta(j\omega)| = \left( \sum_{k=1}^{q} \sum_{l=1}^{q} G_{kl}(j\omega) \right)^{-1}$, the uncertainty

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CONCLUSION

The conclusion section should provide a summary of the main findings and implications of the research. It should highlight the key contributions of the study and suggest areas for future research. In this section, you can also discuss the practical applications of the research and its relevance to the field.

REFERENCES

This section lists all the sources that were cited in the text. Each reference should include the author(s), title, journal, volume, issue, and page numbers. The format of the references should follow the guidelines specified by the publication or the citation style required by your institution.

The conclusion section should conclude the research paper and provide a summary of the main findings. It should also include suggestions for future research and the implications of the study.