Argument Conditions for Hurwitz and Schur Stable
Polynomials and Robust Stability Problem

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ABSTRACT

In the following it is shown that the argument of a Hurwitz polynomial $f(s) = h(s^2) + sg(s^2)$ in the frequency domain $f(j\omega) = h(-\omega^2) + j\omega g(\omega^2)$ as well as the argument of the modified functions $f_1^*(\omega) = h(-\omega^2) + jg(\omega^2)$ and $f_2^*(\omega) = h(-\omega^2) + j\omega^2 g(\omega^2)$ are monotonically increasing functions of $\omega$ whereby $\omega$ varies between 0 and $\infty$.

Also it is shown that given a Schur polynomial $f(z) = h(z) + g(z)$, where $h(z)$ and $g(z)$ are the symmetric and the antisymmetric parts of $f(z)$ respectively, the same monotony property can be obtained for an auxiliary function $f^*(\theta)$ defined as $f(e^{j\theta}) = 2e^{jn\theta/2}f^*(\theta)$. The argument of $f^*(\theta) = h^*(\theta) + jg^*(\theta)$ as well as the argument of the modified functions

$$f_3^*(\theta) = \frac{h^*(\theta)}{\cos \frac{\theta}{2}} + j\frac{g^*(\theta)}{\sin \frac{\theta}{2}}$$

and

$$f_4^*(\theta) = \frac{h^*(\theta)}{\sin \frac{\theta}{2}} + j\frac{g^*(\theta)}{\cos \frac{\theta}{2}}$$

are monotonically increasing functions of $\theta$ whereby $\theta$ varies between 0 and $\pi$.

These results can be directly applied to the robust stability problem.

1. INTRODUCTION

In the theory of robust stability frequency domain considerations are becoming more and more important [1]-[7]. Motivated by some new results on the stabilization of interval plants [8] we investigate in this report the monotony property of Hurwitz and Schur polynomials in the frequency domain.

It will be obvious, that from the stability of some special families of polynomials the robust stability problems can be simplified significantly. More precisely, instead
of edge stability tests using \([9]-[12]\) some extreme point results i.e. corner stability can be used.

2. MAIN RESULT

In this section we put together all monotony results concerning single polynomials. The first result is the one already used in the "principle of argument".

**Theorem 1.** For a Hurwitz stable polynomial \(f(s) = h(s^2) + sg(s^2), \ f(j\omega) = h(-\omega^2) + j\omega g(-\omega^2)\) has a monotonically increasing argument as \(\omega\) increases from 0 to \(\infty\).

In the second result the imaginary part of the polynomial \(f(s)\) along the imaginary axis is modified by \(1/\omega\) and \(\omega\) respectively.

**Theorem 2.** For a Hurwitz stable polynomial \(f(s) = h(s^2) + sg(s^2)\),

\[ f_1^*(\omega) = h(-\omega^2) + jg(-\omega^2) \quad \text{and} \quad f_2^*(\omega) = h(-\omega^2) + j\omega^2 g(-\omega^2) \]

have monotonically increasing arguments as \(\omega\) increases from 0 to \(\infty\).

Now we shall show that corresponding results are also true for Schur polynomials. The monotony of the argument will be considered with respect to rotating coordinates as was used already in [3]. Theorem 3 is the counterpart of theorem 1 for Schur polynomials.

**Theorem 3.** For a Schur stable polynomial \(f(z) = h(z) + g(z)\) of degree \(n\) where \(h(z)\) and \(g(z)\) are the symmetric and the antisymmetric parts of \(f(z)\) respectively, \(f^*(\theta) = h^*(\theta) + jg^*(\theta)\) has a monotonically increasing argument as \(\theta\) increases from 0 to \(\pi\), whereby \(h^*, g^*\) are given by \(f(e^{j\theta}) = 2e^{j\theta} h^*(h^* + jg^*)\).

Starting from theorem 3, modifications of the symmetric and antisymmetric parts of \(f(z)\) can be done in such a way that the monotony property is preserved.

**Theorem 4.** For a Schur polynomial \(f(z) = h(z) + g(z)\), the arguments of the two modified functions

\[ f_3^*(\theta) = \frac{h^*(\theta)}{\cos \frac{\theta}{2}} + j \frac{g^*(\theta)}{\sin \frac{\theta}{2}} \quad \text{and} \quad f_4^*(\theta) = \frac{h^*(\theta)}{\sin \frac{\theta}{2}} + j \frac{g^*(\theta)}{\cos \frac{\theta}{2}} \]

are monotonically increasing as \(\theta\) increases from 0 to \(\theta\) whereby \(h(z), g(z), h^*(\theta), g^*(\theta)\) are as defined in theorem 3.

As \(s = \frac{z-1}{z+1}\) is a bilinear transformation which transforms the \(j\omega\) axis into the unit circle \(z = e^{j\theta}\), it is obvious that \(\omega\) will correspond to \(\tan \frac{\theta}{2}\).
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3. PROOF OF THE RESULTS

To prove the above four theorems we use mathematical induction. We shall prove that if the result is true for polynomial of degree \( n \), it is also true for \( n + 2 \). This covers the general case of complex conjugate or real zeros. By induction from \( n \) to \( n + 1 \) only real zeros can be added which is a special case. To complete the induction it is necessary to prove the results for two low degree polynomials.

**Proof.** of theorem 1.

Although it is well known that theorem 1 can be proved easily using geometrical considerations, we shall use an algebraic proof because of the methodical relation to the proofs of the modified functions.

Let \( f(j\omega) = h(-\omega^2) + j\omega g(-\omega^2) \). To simplify the formulas we suppress the arguments of the polynomials \( h \) and \( g \) respectively. We denote the derivations of \( h(\lambda) \) and \( g(\lambda) \) w.r.t. \( \lambda \) by \( h' \) and \( g' \) respectively.

Because of the almost everywhere monotony of the tangent function the argument of \( f(j\omega) \) is an increasing function of \( \omega \) if and only if \( \frac{d}{d\omega} \left( \frac{\omega g}{h} \right) \) is positive almost everywhere. Assume this monotony condition is fulfilled for Hurwitz stable polynomial \( f(s) \) of degree \( n \):

\[
\frac{d}{d\omega} \left( \frac{\omega g}{h} \right) = \frac{2\omega^2(gh' - hg') + hg}{h} > 0 \iff 2\omega^2(gh' - hg') + hg > 0 \tag{1}
\]

Consider \( f_{n+2}(s) = (s^2 + as + b)f(s) \) where \( a, b > 0 \). Then the new polynomial is also Hurwitz stable and we obtain

\[
f_{n+2}(j\omega) = (b - \omega^2 + aj\omega)(h + j\omega g) = bh - \omega^2h - aw^2g + j\omega(ah + bg - \omega^2g) \tag{2}
\]

For \( f_{n+2}(j\omega) \) we determine the derivative corresponding to equation (1).

\[
\frac{d}{d\omega} \left( \frac{\omega h + bg - \omega^2g}{bh - \omega^2h - aw^2g} \right) = \omega \frac{d}{d\omega} \left( \frac{ah + bg - \omega^2g}{bh - \omega^2h - aw^2g} \right) + \frac{ah + bg - \omega^2g}{bh - \omega^2h - aw^2g} \tag{3}
\]

and

\[
\frac{d}{d\omega} \left( \frac{ah + bg - \omega^2g}{bh - \omega^2h - aw^2g} \right) = -2\omega \left[ (bh - \omega^2h - aw^2g)(ah' + bg' - \omega^2g' + g) - (ah + bg - \omega^2g)(bh' - \omega^2h' - aw^2g' + h + ag) \right] / (bh - \omega^2h - aw^2g)^2
\]

\[
= 2\omega \left[ (\omega^2 - b)^2 + a^2 \omega^2 \right] \frac{(gh' - hg') + a(h^2 + ahg + bg^2)}{(bh - \omega^2h - aw^2g)^2} \tag{4}
\]

Substituting from (4) in (3) we get

\[
\frac{d}{d\omega} \left( \frac{\omega h + bg - \omega^2g}{bh - \omega^2h - aw^2g} \right) = \frac{[(\omega^2 - b)^2 + a^2 \omega^2] \left[ 2\omega^2(gh' - hg') + hg \right] + a(\omega^2 + b)(h^2 + \omega^2g^2)}{(bh - \omega^2h - aw^2g)^2} \tag{5}
\]
From (1) and the positivity of \( a, b \) all terms in (5) are positive. Therefore, if the theorem is true for \( n \), it is also true for \( n + 2 \).

To complete the proof of the theorem we need the positivity for \( n = 1 \) and \( n = 2 \).

For \( n = 1 \) we have \( f(s) = s + a \) and \( f(j\omega) = a + j\omega \). From Hurwitz stability of \( f(s) \) it follows \( a > 0 \) and the monotony condition

\[
\frac{d}{d\omega} \left( \frac{\omega}{a} \right) = \frac{1}{a} > 0
\]  

(6)

For \( n = 2 \) we have \( f(s) = s^2 + as + b \) and \( f(j\omega) = b - \omega^2 + j\omega a \). From Hurwitz stability of \( f(s) \) it follows \( a > 0, b > 0 \), and the monotony condition

\[
\frac{d}{d\omega} \left( \frac{\omega a}{b - \omega^2} \right) = \frac{a(b - \omega^2) + 2\omega a^2}{(b - \omega^2)^2} = \frac{a(\omega^2 + b)}{(b - \omega^2)^2} > 0
\]  

(7)

Therefore, from (6) and (7) the theorem is true for \( n = 1, n = 2 \); then from (5) it is true for all \( n = 1, 2, 3, \ldots n \).

Q.E.D.

**Proof.** of theorem 2.

Theorem 2 consists of two parts. Consider first \( f_1(j\omega) = h(-\omega^2) + jg(-\omega^2) \).

The condition for monotony of argument of \( f_1(j\omega) \) as function of \( \omega \) is

\[
\frac{d}{d\omega} \left( \frac{g}{h} \right) = \frac{(hg' - gh')(2\omega)}{h^2} > 0 \iff 2\omega(g h' - h g') > 0
\]  

(8)

But because (1) holds for all stable \( f(s) \), (8) is obviously fulfilled for \( h g < 0 \). Therefore, we have to prove the theorem only for \( h g > 0 \). We use again induction.

Assume that (8) is true for \( f(s) \) of degree \( n \), then for \( n + 2 \) we get from (2) and (4)

\[
\frac{d}{d\omega} \left( \frac{a h + b g - \omega^2 g}{b h - \omega^2 h - a \omega^2 g} \right) = \frac{2\omega [(\omega^2 - b)^2 + a^2 \omega^2] (g h' - h g') + 2\omega a (h^2 + a h g + b g g')}{(b h - \omega^2 h - a \omega^2 g)^2}
\]  

(9)

With \( \omega > 0, a > 0, b > 0 \) and using (8) all terms are positive if \( h g > 0 \). This proves the first part of the induction.

To close the induction loop we need to prove two starting points. For \( n = 2 \) we obtain for stable \( f(s) \):

\[
f(s) = s^2 + as + b, \quad f(j\omega) = b - \omega^2 + j\omega a \quad \text{and} \quad a > 0, b > 0.
\]

For \( f_2^1(\omega) \) the monotony of the argument is given by

\[
\frac{d}{d\omega} \left( \frac{a}{b - \omega^2} \right) = \frac{2a\omega}{(b - \omega^2)^2} > 0
\]  

(10)

For \( n = 3 \) we obtain \( f(s) = s^3 + as^2 + bs + c \), \( f(j\omega) = c - a \omega^2 + j\omega (b - \omega^2) \) and the stability conditions \( a > 0, ab - c > 0, c > 0 \). It follows

\[
\frac{d}{d\omega} \left( \frac{b - \omega^2}{c - a \omega^2} \right) = \frac{2\omega(ab - c)}{(c - a\omega^2)^2} > 0
\]  

(11)
Therefore, from (10) and (11) the first part of the theorem is true for \( n = 2, n = 3 \), then using (9) it is true for all \( n = 2, 3, 4, \ldots \) Here we did not use \( n = 1 \) because \( f_1^2(\omega) \) is independent of \( \omega \).

We prove now the second part of theorem 2.
Consider \( f_2^2(\omega) = h + j\omega^2g \). Then
\[
\frac{d}{d\omega} \left( \frac{\omega^2g}{h} \right) = \omega^2 \frac{d}{d\omega} \left( \frac{g}{h} \right) + 2\omega \frac{g}{h} > 0 \quad \iff \quad \omega^2(gh' - hg') + hg > 0 \quad (12)
\]
We use again induction to prove that \( \frac{d}{d\omega} \left( \frac{\omega^2g}{h} \right) > 0 \) holds also for extended \( f_2^2(\omega) \) of order \( n + 2 \) from (2).
\[
\frac{d}{d\omega} \left( \frac{\omega^2(ah + bg - \omega^2g)}{bh - \omega^2h - aw^2g} \right) = \omega^2 \frac{d}{d\omega} \left( \frac{ah + bg - \omega^2g}{bh - \omega^2h - aw^2g} \right) + 2\omega \frac{ah + bg - \omega^2g}{bh - \omega^2h - aw^2g} \quad (13)
\]
Substituting from (4)
\[
\frac{d}{d\omega} \left( \frac{\omega^2(ah + bg - \omega^2g)}{bh - \omega^2h - aw^2g} \right) = 2\omega \left\{ 2\omega^2b + a^2\omega^2 \right\} \left\{ (\omega^2(gh' - hg') + hg) \right\} + \frac{abh^2 + aw^4g^2}{(bh - \omega^2h - aw^2g)^2} \quad (13)
\]
Using (12) positivity of (13) follows. Therefore monotony increase of the argument of \( f_2^2(\omega) \) is true for degree \( n + 2 \) if it is true for degree \( n \).
To complete the induction we prove (12) for two starting points. For \( n = 1 \):
\[
\frac{d}{d\omega} \left( \frac{\omega^2g}{h} \right) = \frac{d}{d\omega} \left( \frac{\omega^2}{a} \right) = 2\omega \frac{a}{a} > 0 \quad (14)
\]
and for \( n = 2 \):
\[
\frac{d}{d\omega} \left( \frac{\omega^2g}{h} \right) = \frac{d}{d\omega} \left( \frac{\omega^2a}{b - \omega^3} \right) = \frac{2a\omega b}{(b - \omega^3)^2} > 0 \quad (15)
\]
Therefore from (14) and (15) the \( f_2^2 \) part of the theorem is true for \( n = 1, n = 2 \). Then with (13) it is true for \( n = 1, 2, 3, \ldots n \). Q.E.D.

Proof. of theorem 3
Now, we prove the Schur counterpart of theorem 1.
Let \( f(z) = h(z) + g(z) \) be a Schur stable polynomial where \( h(z) \) and \( g(z) \) are the symmetric and antisymmetric parts respectively.
\[
\begin{align*}
  h(z) &= \frac{1}{2} \left[ f(z) + z^n f \left( \frac{1}{z} \right) \right] \\
  g(z) &= \frac{1}{2} \left[ f(z) - z^n f \left( \frac{1}{z} \right) \right]
\end{align*}
\]
With \( \alpha_i, \beta_i \) as the parameters of the symmetric and antisymmetric parts respectively used in [13]
\[
\begin{align*}
  h(z) &= \alpha_0 z^n + \alpha_1 z^{n-1} + \cdots + \alpha_1 z + \alpha_0 \\
  g(z) &= \beta_0 z^n + \beta_1 z^{n-1} + \cdots - \beta_1 z - \beta_0
\end{align*}
\]
We obtain along the unit circle $z = e^{i\theta}$

$$f(e^{i\theta}) = 2e^{jn\theta/2}[h^*(\theta) + jg^*(\theta)]$$  \hspace{1cm} (16)

This is a decomposition of $f(e^{i\theta})$ into two orthogonal parts w.r.t. rotating coordinates. For $n$ even, i.e. $n = 2\nu$, we get

$$h^*(\theta) = \alpha_0 \cos\nu\theta + \alpha_1 \cos(\nu - 1)\theta + \ldots + \alpha_{\nu-1} \cos\frac{\theta}{2}$$

$$g^*(\theta) = \beta_0 \sin\nu\theta + \beta_1 \sin(\nu - 1)\theta + \ldots + \beta_{\nu-1} \sin\frac{\theta}{2}$$

For $n$ odd, i.e. $n = 2\nu - 1$, we get

$$h^*(\theta) = \alpha_0 \cos\left(\nu - \frac{1}{2}\right)\theta + \alpha_1 \cos\left(\nu - \frac{3}{2}\right)\theta + \ldots + \alpha_{\nu-1} \cos\frac{\theta}{2}$$

$$g^*(\theta) = \beta_0 \sin\left(\nu - \frac{1}{2}\right)\theta + \beta_1 \sin\left(\nu - \frac{3}{2}\right)\theta + \ldots + \beta_{\nu-1} \sin\frac{\theta}{2}$$

Now we shall use induction to prove that $\frac{d}{d\theta}\left(\frac{g^*}{h^*}\right) > 0$. Assume first

$$\frac{d}{d\theta}\left(\frac{g^*}{h^*}\right) = \frac{h^* h^* - g^* h^*}{(h^*)^3} > 0 \iff h^* g^* - g^* h^* > 0$$  \hspace{1cm} (17)

the positivity of (17) for $f(z)$ of degree $n$. Now we extend the Schur polynomial $f(z)$ to a new Schur polynomial $f_{n+2}(z)$ of degree $n + 2$

$$f_{n+2}(z) = (\alpha_0 z^2 + \alpha_1 z + \alpha_0 + \beta_0 z^2 + \beta_1 z - \beta_0) f(z)$$

Here

$$\alpha_0 > 0, \quad \beta_0 > 0, \quad 2\alpha_0 > |\alpha_1|$$  \hspace{1cm} (18)

are stability conditions for the second degree polynomial.

The associated decomposition into orthogonal components is

$$h^*_{n+2} + jg^*_{n+2} = \left(\alpha_0 \cos\theta + \frac{\alpha_1}{2} + j\beta_0 \sin\theta\right)\left(h^* + jg^*\right)$$

$$= \alpha_0 h^* \cos\theta + \frac{\alpha_1}{2} h^* - \beta_0 g^* \sin\theta + j \left(\beta_0 h^* \sin\theta + \alpha_0 g^* \cos\theta + \frac{\alpha_1}{2} g^*\right)$$  \hspace{1cm} (19)

The monotony of the argument of $f_{n+2}(z)$ is equivalent to positivity of

$$\frac{d}{d\theta}\left(\frac{g^*_{n+2}}{h^*_{n+2}}\right) = \frac{d}{d\theta}\left(\frac{\beta_0 h^* \sin\theta + \alpha_0 g^* \cos\theta + \frac{\alpha_1}{2} g^*}{\alpha_0 h^* \cos\theta + \frac{\alpha_1}{2} h^* - \beta_0 g^* \sin\theta}\right)$$

$$= \beta_0 \left(\alpha_0 + \frac{\alpha_1}{2}\cos\theta\right) \left(h^{*2} + g^{*2}\right) + \left[\left(\alpha_0 \cos\theta + \frac{\alpha_1}{2}\right)^2 + \beta_0^2 \sin^2\theta\right] (h^* g^* - g^* h^*) > 0$$  \hspace{1cm} (20)

Assuming positivity of (17) and using (18) the positivity of (20) is obvious. Therefore if the theorem is true for $n$, it is also true for $n + 2$. 


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To complete the induction we need to prove (17) for \( n = 1 \) and \( n = 2 \). For \( n = 1 \) we have \( f(z) = z + a \) with the stability condition \( |a| < 1 \). The decomposition to the symmetric/antisymmetric parts gives

\[
h(z) = \alpha_0 z + \alpha_0, \quad g(z) = \beta_0 z - \beta_0
\]

with the stability condition \( |a| < 1 \iff \alpha_0 > 0, \beta_0 > 0 \)

The two orthogonal parts are

\[
h^* = \alpha_0 \cos \left( \frac{\theta}{2} \right), \quad g^* = \beta_0 \sin \left( \frac{\theta}{2} \right)
\]

and the monotony condition delivers directly

\[
\frac{d}{d\theta} \left( \frac{g^*}{h^*} \right) = \frac{d}{d\theta} \left( \frac{\beta_0 \sin \left( \frac{\theta}{2} \right)}{\alpha_0 \cos \left( \frac{\theta}{2} \right)} \right) = \frac{1}{2} \alpha_0 \beta_0 > 0
\]

(21)

For \( n = 2 \) we have \( f(z) = z^2 + az + b \) and the stability conditions:

\[
1 + a + b > 0, \quad 1 - a + b > 0, \quad b < 1,
\]

The symmetric/antisymmetric parts are

\[
h(z) = \alpha_0 z^2 + \alpha_1 z + \alpha_0, \quad g(z) = \beta_0 z^2 - \beta_0
\]

with

\[
\alpha_0 = \frac{1 + b}{2}, \quad \alpha_1 = a, \quad \beta_0 = \frac{1 - b}{2}
\]

and the associated stability conditions:

\[
\alpha_0 > 0, \quad \beta_0 > 0, \quad 2 \alpha_0 > |\alpha_1|
\]

Decomposition to orthogonal parts delivers

\[
h^* = \alpha_0 \cos \theta + \frac{\alpha_1}{2} \quad g^* = \beta_0 \sin \theta
\]

where \( 0 \leq \theta \leq \pi \).

From the monotony condition

\[
\frac{d}{d\theta} \left( \frac{g^*}{h^*} \right) = \frac{d}{d\theta} \left( \frac{\beta_0 \sin \theta}{\alpha_0 \cos \theta + \frac{\alpha_1}{2}} \right) = \frac{\beta_0 \left( \alpha_0 + \frac{\alpha_1}{2} \cos \theta \right)}{\left( \alpha_0 \cos \theta + \frac{\alpha_1}{2} \right)^2}
\]

(22)

and using the stability conditions we obtain the positivity of (22).

Therefore from (21) and (22), theorem 3 is true for \( n = 1, n = 2 \), then from (20) it is true for all \( n = 1, 2, 3, \ldots \). This result can be also proved geometrically. Q.E.D.
Proof. of theorem 4.
Consider first
\[ f_3'(\theta) = \frac{h^* (\theta)}{\cos \frac{\theta}{2}} + j g^*(\theta) \frac{\sin \frac{\theta}{2}}{\sin \frac{\theta}{2}}. \]
The monotony condition w.r.t. \( \theta \) is the positivity of
\[
\frac{d}{d\theta} \left( \frac{g^* \cos \left( \frac{\theta}{2} \right)}{h^* \sin \left( \frac{\theta}{2} \right)} \right)
= \frac{h^* \sin \left( \frac{\theta}{2} \right) \left[ g^* \cos \left( \frac{\theta}{2} \right) - \frac{1}{2} g^* \sin \left( \frac{\theta}{2} \right) - g^* \cos \left( \frac{\theta}{2} \right) \left( h^* \sin \left( \frac{\theta}{2} \right) + \frac{1}{2} h^* \cos \left( \frac{\theta}{2} \right) \right) \right]}{\left( h^* \sin \left( \frac{\theta}{2} \right) \right)^2}
= \frac{(h^* g^* - g^* h^*) \sin \theta - h^* g^*}{2 \left( h^* \sin \left( \frac{\theta}{2} \right) \right)^2} > 0 \quad (23)
\]
Considering (17) it is obvious that (23) is positive for all \( h^* g^* < 0 \). Therefore we need to prove only the case \( h^* g^* > 0 \). For this we use induction once more.

Consider the monotony condition for the extended polynomial \( f_{n+2}(z) \) with the decomposition (19):
For \( h^* g^* > 0 \) we use induction
\[
\frac{d}{d\theta} \left( \frac{g_{n+2}^* \cos \frac{\theta}{2}}{h_{n+2}^* \sin \frac{\theta}{2}} \right) = \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}}.
\]
\[
\beta_0 \left( \alpha_0 + \frac{\theta}{2} \cos \theta \right) \left( h_{n+2}^* + g_{n+2}^* \right) + \left[ \left( \alpha_0 \cos \theta + \frac{\theta}{2} \right)^2 + \beta_0^2 \sin^2 \theta \right] \left( h_{n+2}^* g_{n+2}^* - g_{n+2}^* h_{n+2}^* \right)
+ \frac{\beta_0 h^* \sin \theta + \alpha_0 g^* \cos \theta + \frac{\theta}{2} g^*}{\alpha_0 h^* \cos \theta + \frac{\theta}{2} h^* - \beta_0 g^* \sin \theta} \cdot \frac{-1}{2 \sin^2 \frac{\theta}{2}}
\]
This gives
\[
\frac{1}{2} \left[ \left( \alpha_0 \cos \theta + \frac{\theta}{2} \right)^2 + \beta_0^2 \sin^2 \theta \right] \sin \theta \left( h_{n+2}^* g_{n+2}^* - g_{n+2}^* h_{n+2}^* \right) + \beta_0 h_{n+2}^* g_{n+2}^* \sin^2 \theta
+ \frac{1}{2} \beta_0 \sin \theta \left( \alpha_0 - \frac{\theta}{2} \right) (1 - \cos \theta) h_{n+2}^* + \frac{1}{2} \beta_0 \sin \theta \left( \alpha_0 + \frac{\theta}{2} \right) (1 + \cos \theta) g_{n+2}^* > 0
\quad (24)
\]
Therefore if theorem 4 is true for \( n \) it is also true for \( n+2 \).
We prove now the theorem for \( n = 2 \) and \( n = 3 \). For \( n = 2 \) we obtain
\[
h^* = \alpha_0 \cos \theta + \frac{\theta}{2}, \quad g^* = \beta_0 \sin \theta
\]
and the monotony condition
\[
\frac{d}{d\theta} \left( \frac{g^* \cos \left( \frac{\theta}{2} \right)}{h^* \sin \left( \frac{\theta}{2} \right)} \right) = \frac{\cos \left( \frac{\theta}{2} \right) \cdot \beta_0 \left( \alpha_0 + \frac{\theta}{2} \cos \theta \right)}{\sin \left( \frac{\theta}{2} \right) \left( \alpha_0 \cos \theta + \frac{\theta}{2} \right)^2} + \frac{\beta_0 \sin \theta}{\alpha_0 \cos \theta + \frac{\theta}{2}} \cdot \frac{-1}{2 \sin^2 \frac{\theta}{2}}
\]
This gives
\[ \frac{1}{2} \beta_0 (1 - \cos \theta) \left( \alpha_0 - \frac{\alpha_1}{2} \right) \sin \theta > 0 \] (25)
which is obviously positive using the stability condition.
For \( n = 3 \) we have
\[ f(z) = z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0 \]
The symmetric/antisymmetric parts are
\[ h = \alpha_0 z^3 + \alpha_1 z^2 + \alpha_1 z + \alpha_0 \]
and
\[ g = \beta_0 z^3 + \beta_1 z^2 - \beta_1 z - \beta_0 \]
where
\[ \alpha_0 = \frac{1 + c}{2}, \quad \alpha_1 = \frac{a + b}{2}, \quad \beta_0 = \frac{1 - c}{2}, \quad \beta_1 = \frac{a - b}{2} \]
Therefore
\[ h^* = \alpha_0 \cos \frac{3}{2} \theta + \alpha_1 \cos \frac{\theta}{2} \]
\[ g^* = \beta_0 \sin \frac{3}{2} \theta + \beta_1 \sin \frac{\theta}{2} \]
and the stability conditions are:
\[
\begin{align*}
1 + a + b + c > 0 & \iff \alpha_0 + \alpha_1 > 0 \\
1 - a + b - c > 0 & \iff \beta_0 - \beta_1 > 0 \\
1 - c^2 - b + ac > 0 & \iff 2\alpha_0\beta_0 + \alpha_0\beta_1 - \alpha_1\beta_0 > 0 \\
1 + c > 0 & \iff \alpha_0 > 0 \\
1 - c > 0 & \iff \beta_0 > 0
\end{align*}
\]
From the modified decomposition
\[
\frac{g^*}{\sin \frac{\theta}{2}} = \beta_0 \left( \frac{3 - 4 \sin^2 \frac{\theta}{2}}{2} \right) + \beta_1 = \beta_0 (2 \cos \theta + 1) + \beta_1
\]
\[
\frac{h^*}{\cos \frac{\theta}{2}} = \alpha_0 \left( \frac{4 \cos^2 \frac{\theta}{2} - 3}{2} \right) + \alpha_1 = \alpha_0 (2 \cos \theta - 1) + \alpha_1
\]
follows for monotony of
\[
\frac{d}{d\theta} \left( \frac{g^* \cos \frac{\theta}{2}}{h^* \sin \frac{\theta}{2}} \right) = \frac{d}{d\theta} \left( \frac{\beta_0 (2 \cos \theta + 1) \beta_1}{\alpha_0 (2 \cos \theta - 1) + \alpha_1} \right)
\]
This gives
\[
\begin{align*}
&[\alpha_0 (2 \cos \theta - 1) + \alpha_1] (-2\beta_0 \sin \theta) - [\beta_0 (2 \cos \theta + 1) + \beta_1] (-2\alpha_0 \cos \theta) \\
&= 4\alpha_0\beta_0 \sin \theta + 2 \sin \theta (\alpha_0\beta_1 - \alpha_1\beta_0) \\
&= 4 \sin \theta \left( \alpha_0\beta_0 + \frac{1}{2} (\alpha_0\beta_1 - \alpha_1\beta_0) \right) > 0
\end{align*}
\] (26)
whereby for positivity the stability conditions are used.
From (25), (26) theorem 4 is true for $n = 2, 3$, then it is also true for all
$n = 2, 3, 4, 5, \ldots$

Consider next

$$ f_i^*(\theta) = \frac{h^*(\theta)}{\sin \frac{\theta}{2}} + j \frac{g^*(\theta)}{\cos \frac{\theta}{2}} $$

and the associated monotony condition

$$ \frac{d}{d\theta} \left( \frac{g^* \sin \frac{\theta}{2}}{h^* \cos \frac{\theta}{2}} \right) = \frac{(h^* g^{*-} - g^* h^{*-}) \sin \theta + h^* g^*}{2 \left( h^* \cos \frac{\theta}{2} \right)^2} \tag{27} $$

Considering (17) the positivity of (27) follows immediately for $h^* g^* > 0$. Therefore we need to prove the theorem only for $h^* g^* < 0$. Using again induction we assume the positivity of (27) and calculate the monotony condition for polynomial of order
$n + 2$.

$$ \frac{d}{d\theta} \left( \frac{g_{n+2}^* \sin \frac{\theta}{2}}{h_{n+2}^* \cos \frac{\theta}{2}} \right) = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \cdot \beta_0 \left( \alpha_0 + \frac{\alpha_1}{2} \cos \theta \right) (h^* g^{+2} + g^* h^{+2}) + \left[ \left( \alpha_0 \cos \theta + \frac{\alpha_1}{2} \right)^2 + \beta_0^2 \sin^2 \theta \right] \left( \frac{h^* g^{*-} - g^* h^{*-}}{\cos \frac{\theta}{2}} \right)^2 \right] $$

$$ + \frac{1}{\alpha_0 h^* \cos \theta + \frac{\alpha_1}{2} h^* - \beta_0 g^* \sin \theta} \cdot \frac{1}{2 \cos^2 \frac{\theta}{2}} \tag{28} $$

This gives

$$ \left[ \left( \alpha_0 \cos \theta + \frac{\alpha_1}{2} \right)^2 + \beta_0^2 \sin^2 \theta \right] \left[ \frac{1}{2} \sin \theta \left( h^* g^{+2} - g^* h^{+2} \right) + \frac{1}{2} h^* g^* \right] - \beta_0 h^* g^* \sin^2 \theta $$

$$ + \frac{1}{2} \beta_0 \sin \theta \left( \alpha_0 + \frac{\alpha_1}{2} \right) (1 + \cos \theta) h^* g^* + \frac{1}{2} \beta_0 \sin \theta \left( \alpha_0 - \frac{\alpha_1}{2} \right) (1 - \cos \theta) g^* \sin^2 \theta > 0 $$

The positivity of (28) follows from (27) which was assumed.

The two special cases $n = 1$ and $n = 2$ must be checked. For $n = 1$ we obtain

$$ \frac{d}{d\theta} \left( \frac{g^* \sin \frac{\theta}{2}}{h^* \cos \frac{\theta}{2}} \right) = \frac{d}{d\theta} \left( \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \cdot \frac{\beta_0 \sin \frac{\theta}{2}}{\alpha_0 \cos \frac{\theta}{2}} \right) $$

$$ = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \cdot \frac{\alpha_0 \beta_0}{2} + \frac{\beta_0 \sin \frac{\theta}{2}}{\alpha_0 \cos \frac{\theta}{2}} \cdot \frac{1}{2 \cos^2 \frac{\theta}{2}} $$

$$ = \frac{\alpha_0 \beta_0 \sin \theta}{2 \alpha_0^2 \cos^4 \frac{\theta}{2}} > 0 \tag{29} $$

For $n = 2$ follows

$$ \frac{d}{d\theta} \left( \frac{g^* \sin \frac{\theta}{2}}{h^* \cos \frac{\theta}{2}} \right) = \frac{d}{d\theta} \left( \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \cdot \frac{\beta_0 \sin \frac{\theta}{2}}{\alpha_0 \cos \theta + \frac{\alpha_1}{2}} \right) $$

$$ = \frac{\beta_0 \sin \theta (1 + \cos \theta) \left( \alpha_0 + \frac{\alpha_1}{2} \right)}{2 \left( \alpha_0 \cos \theta + \frac{\alpha_1}{2} \right) \cos^2 \frac{\theta}{2}} > 0 \tag{30} $$
ARGUMENT CONDITIONS

From (29), (30) theorem 4 is true for \( n = 1, 2 \), then with (28) it is also true for all \( n = 1, 2, 3, \ldots \)

Q.E.D.

Remark 1. Prof. N. Bose has shown in a personal communication that the monotony of the argument can be proved using results in network theory. See e.g. [14] and [15].

4. APPLICATION TO ROBUST STABILITY PROBLEM

4.1. Hurwitz stability

Consider the edge given by

\[
f(s, \lambda) = f_0(s) + \lambda (as + b)k(s) \quad \lambda \in [0, 1]
\]

where \( k(s) \) is an even or odd polynomial of degree \( < n - 1 \), and \( f_0(s) \) is a polynomial of degree \( n \).

Let \( f_0(s) \) and \( f_0(s) + (as + b)k(s) \) be two Hurwitz stable polynomials, then the stability of \( f(s, \lambda) \) can be obtained as follows:

Instead of using the value set of \( f(s) \) directly we use the value set in the modified functions \( f^*_1 \) or \( f^*_2 \) for even or odd \( k(s) \) respectively. It can be easily shown that then the value set is a straight line with constant slope.

(a) \( k(s) \) is an even polynomial. Then

\[
f(s, \lambda) = f_0(s) + \lambda (as + b)h(s^2)
\]

and

\[
f^*_1(\omega, \lambda) = f^*_0(\omega) + \lambda(b + ja)h(-\omega^2)
\]

For \( \lambda \in [0, 1] \) the value set of \( f^*_1(\omega, \lambda) \) is a straight line with the constant slope \( \frac{a}{b} \).

(b) \( k(s) \) is an odd polynomial. Then

\[
f(s, \lambda) = f_0(s) + \lambda (as + b)sg(s^2)
\]

and the modified function

\[
f^*_2(\omega, \lambda) = f^*_0(\omega) + \lambda(-a + jb)\omega^2g(-\omega^2)
\]

For \( \lambda \in [0, 1] \) the value set of \( f^*_2(\omega, \lambda) \) is a straight line with the constant slope \( \frac{-b}{a} \).

Because of the assumed stability of the extreme polynomials (corners for \( \lambda = 0 \) and \( \lambda = 1 \)) we have the monotony of the argument. From the constant slope of the edge it follows immediately that the situation shown in Fig. 1 can not occur, where \( \Delta f^*_0 \) corresponds to \( (as + b)k(s) \) for one of the cases discussed before. With the same arguments as used in [2] we can exclude the origin from the edge. Hence
the robust Hurwitz stability of the edge follows just from the stability of the corner polynomials.

This result was proved in [8] using complex polynomials. The same is valid for $f_0(s) + \lambda(a\bar{s} + b)s'k(s)$ where degree $k(s) + r < n - 1$.

Using this result robust Hurwitz stability for control systems with first order compensator (LEAD/LAG) and interval plants can be very easily checked. For robust stability only 8 corners must be Hurwitz stable. For complete discussion of this subject see [8] and [5].

4.2. Schur stability

Similar to the above we consider the edge

$$f(z, \lambda) = f_0(z) + \lambda(az + b)k(z)$$

where $k(z)$ is symmetric or antisymmetric polynomial of degree $n - 1$, $\lambda \in [0, 1]$ and $f_0(z)$ is a Schur polynomial of degree $n$. The stability of the edge $f(\lambda, z)$ can be obtained in a similar way as for Hurwitz stability considering the stability of the corner polynomials and applying theorem 4 as follows:

We decompose $az + b$ into its symmetric and antisymmetric parts

$$\alpha_0 z + \alpha_0 + \beta_0 z - \beta_0$$

Now for $k(z)$ symmetric, i.e. $k^*(\theta) = h^*$ we obtain

$$f^*(\theta) = f_0^*(\theta) + \lambda \left[ \alpha_0 h^* \cos \frac{\theta}{2} + j \beta_0 h^* \sin \frac{\theta}{2} \right]$$
and the modified function $f_3^*$ is

$$f_3^* = f_{03}^* + \lambda [\alpha_0 h^* + j\beta_0 h^*]$$

For this modified function the edge has a constant slope. Similarly for $k(z)$ antisymmetric, i.e. $k^*(\theta) = jg^*$ we obtain

$$f^*(\theta) = f_0^*(\theta) + \lambda \left[ j\alpha_0 g^* \cos \frac{\theta}{2} - \beta_0 g^* \sin \frac{\theta}{2} \right]$$

Therefore using the modified function $f_4^*$ we obtain

$$f_4^* = f_{04}^* + \lambda [\beta_0 g^* + j\alpha_0 g^*]$$

and the slope of this modified edge is constant. Using theorem 4 and same argument as for Hurwitz stability Fig. 1 we have Schur stability of the edge from the stability of the corners of $f(z, \lambda)$, i.e. $f_0(z)$ and $f(z, 1)$.

Also the same is valid for $f_0(z) + \lambda(az + b)(z - 1)^r k(z)$ where degree $k(z) + r = n - 1$.

Similar results can be obtained for Schur stability of control systems with first order compensator.

5. CONCLUSION

It was proved that the argument of

$$f(j\omega) = h + j\omega g, \quad f_1^* = h + jg, \quad f_2^* = h + jg\omega^2$$

for a Hurwitz polynomial $f(s)$ are monotonically increasing. Also for a Schur polynomial $f(z)$, with $f(e^{j\theta}) = 2e^{j\theta/2}f^*$, the arguments of

$$f^* = h^* + jg^*, \quad f_3^* = \frac{h^*}{\cos \frac{\theta}{2}} + j\frac{g^*}{\sin \frac{\theta}{2}},$$

and

$$f_4^* = \frac{h^*}{\sin \frac{\theta}{2}} + j\frac{g^*}{\cos \frac{\theta}{2}}$$

are monotonically increasing.

It was also shown that these results are directly applicable to some special edges, and to the robust stability of control systems with first order compensator.
REFERENCES


