The Generation of Discrete-Time \( q \)-Markov Covers Via Inverse Solution of the Lyapunov Equation

V. Sreeram, P. Agathoklis, and M. Mansour

Abstract—A new method for generating \( q \)-Markov Covers for SISO discrete-time systems is proposed. It is based on computing first the impulse-response Gramian from the Markov parameters and covariances, and then solving the Lyapunov equation inversely to get the system matrices in controllability canonical form. The conditions for existence of \( q \)-Markov Covers are also derived. The method is illustrated by a numerical example and is shown to be computationally simple.

I. INTRODUCTION

Model reduction has attracted considerable attention in the past two decades and many methods have been proposed. One of the most popular and powerful methods, is based on the concept of balanced realization. The balanced realization technique was first proposed in [6] for the reduction of linear continuous systems and is based on forming a realization in which each state is equally controllable and observable and subsequently obtaining the reduced order model by truncating the least controllable and the least observable states. Another popular technique is the \( q \)-Markov Cover method [2], [5], [12]. It is based on retaining in the reduced-order model, \( q \)-Markov parameters and \( q \) output covariances of the original system. Algorithms for obtaining \( q \)-Markov covers have been presented in [12], [13] while in [2] the class of all possible \( q \)-Markov covers has been parameterized. It is shown in [2] that the class includes an infinite set for continuous-time systems, whereas for discrete-time systems the class has only two members. It is also shown that given one \( q \)-Markov cover, one can construct all systems having identical \( q \)-Markov parameters and \( q \) output covariances.

The methods for generating discrete-time \( q \)-Markov Covers for both MIMO and SISO are known and are presented in [2], [5]. As pointed out in [7], however, these techniques require complex procedures and involve a number of steps. In this note, a conceptually simple and computationally efficient method for generating discrete-time \( q \)-Markov covers is presented for SISO systems. The formulas for \( q \)-Markov Covers obtained are essentially identical to that of the formulas derived in [7]. However, the method followed in the paper is entirely different and is based on the inverse solution of the Lyapunov equation. A new method for the solution of Lyapunov equation

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inversely for system matrix (in controllability canonical form) is also presented. The necessary conditions for existence of stable solutions are derived. Using the proposed technique, q-Markov Covers can be expressed explicitly in terms of input parameters namely covariance and Markov parameters of the system.

II. PRELIMINARIES

Consider a stable single-input, single-output discrete system described by the following minimal realization

\[ x(k + 1) = Ax(k) + bu(k) \]  
\[ y(k) = cx(k) \]

where \( x(k) \in \mathbb{R}^n \). A q-Markov cover for such a system is defined as follows

**Definition 1** [2, 5, 12]: The reduced-order model is a q-Markov covariance equivalent realization (q-Markov Cover) of the system equations (2.1) and (2.2) if and only if

\[
(h_r)_i = h_i, \quad i = 1, 2, \ldots, q
\]

and

\[
(R_r)_i = R_i, \quad i = 1, 2, \ldots, q
\]

where

\[
h_i = cA^{i-1}b,
\]

\[
R_i = cA^{i-1}X_r c_r^T,
\]

\[
(h_r)_i = c_rA^{i-1}b_r,
\]

and

\[
(R_r)_i = c_rA^{i-1}X_r c_r^T.
\]

\( X \) and \( X_r \) denote the steady state covariance of the original system and the reduced-order model satisfying the following Lyapunov equations, respectively,

\[
X = AXA^T + BWb^T
\]

\[
X_r = A_rX_rA_r^T + b_rWb_r^T
\]

where \( W \) is the covariance of the input signal and is a scalar for SISO system.

The method proposed in [13] for finding q-Markov cover models is based on the impulse-response Gramian which is defined as follows.

**Definition 2** [9, 10, 13]: The impulse-response Gramian for a stable discrete SISO system is given by

\[
S = \sum_{k=0}^{n} \begin{bmatrix} h_{k+1} & \cdots & h_{k+n} \\ \vdots & \ddots & \vdots \\ h_{k+1} & \cdots & h_{k+n} \end{bmatrix}
\]

where \( h_k = cA^{k-1}b \) is the impulse-response of the system.

**Theorem 2.1** [10]: The impulse-response Gramian, \( S \), satisfies the following Lyapunov equation

\[
\hat{A}S + SA^T - S = -Q = -c^T \hat{c},
\]

where \( \{ \hat{A}, \hat{b}, \hat{c} \} \) is in controllability canonical form [4].

The above theorem is the basis for the solution of Lyapunov equation proposed. Given the matrices \( Q \) and \( \hat{A} \), it is well known [3] that (2.3) can be solved for matrix \( S \). In the next section a method is proposed to compute the matrix \( A \) given the symmetric matrices \( S \) and \( Q \).
where

\[
S_{22} = \begin{bmatrix}
\xi_{22} & \xi_{23} & \cdots & \xi_{2n} \\
\xi_{23} & \xi_{33} & \cdots & \xi_{3n} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{2n} & \xi_{3n} & \cdots & \xi_{nn}
\end{bmatrix}
\]

\[
S_{1n} = [\xi_{1n}, \xi_{2n}, \cdots, \xi_{n-1,n}]^T
\]

\[
Q_{1n} = [\zeta_{1n}, \zeta_{2n}, \cdots, \zeta_{n-1,n}]^T
\]

\[
S_{12} = [\xi_{12}, \xi_{13}, \cdots, \xi_{1n}]^T
\]

Equation (3.4) can be rewritten as

\[
y = [a_n^T \quad a_{n-1}^T] \begin{bmatrix} s_{11} & S_{12}^T \\ S_{12} & S_{22} \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}
\]

Using (3.6) the above equation gives

\[
y = (s_{11} - S_{12}S_{22}^{-1}S_{12})a_n^2 + (S_{1n} - Q_{1n})^T S_{22}^{-1} (S_{1n} - Q_{1n})
\]

Since \( y = s_{nn} - q_{nn} \), (3.2), \( a_n^2 \) can be written as

\[
a_n^2 = \frac{(s_{nn} - q_{nn}) - (S_{1n} - Q_{1n})^T S_{22}^{-1} (S_{1n} - Q_{1n})}{\theta_{11} - S_{12}^T S_{22}^{-1} S_{12}}
\]

In the RHS of the above equation note that the numerator and denominator are scalars. By multiplying the numerator and denominator by \( \det[S_{22}] \) we have

\[
a_n^2 = \frac{\det[S_{22}] \det[(s_{nn} - q_{nn}) - (S_{1n} - Q_{1n})^T S_{22}^{-1} (S_{1n} - Q_{1n})]}{\det[S_{22}] \det[\theta_{11} - S_{12}^T S_{22}^{-1} S_{12}]}
\]

\[
= \frac{\det[S - Q]}{\det[S]}
\]

(3.7)

where the matrices \( S \) and \( S - Q \) in the above equation can be partitioned as follows:

\[
S = \begin{bmatrix}
s_{11} & S_{12}^T \\ S_{12} & S_{22}
\end{bmatrix}
\]

and

\[
S - Q = \hat{A}^T S \hat{A} = \begin{bmatrix}
S_{11} - Q_{11} & (S_{1n} - Q_{1n})^T \\
(S_{1n} - Q_{1n}) & (s_{nn} - q_{nn})
\end{bmatrix}
\]

\[
= \begin{bmatrix}
S_{22} & (S_{1n} - Q_{1n})^T \\
(S_{1n} - Q_{1n}) & (s_{nn} - q_{nn})
\end{bmatrix}
\]

Theorem 3.1: If \( S \) is positive definite \((S > 0)\), \( Q \) is positive semidefinite \((Q \geq 0)\), and \( S - Q \) is at least positive semidefinite, then \( a_n^2 \) is nonnegative and lies in the following range:

\[0 \leq a_n^2 \leq 1.\]

Proof: Since matrix \( S - Q \) is at least positive semidefinite, it follows that \( S \geq S - Q \) and \( \det[S] \geq [S - Q] \). Therefore, \( 0 \leq a_n^2 \leq 1. \)

Remarks:
1) If \( S - Q \) is positive semidefinite, then \( \det[S - Q] = 0. \) This implies that \( a_n^2 = 0 \) and there is a unique inverse solution to the Lyapunov equation. The unknown elements of the unique system matrix are given by

\[
a = \begin{bmatrix}
a_n \\ a_{n-1}
\end{bmatrix} = \begin{bmatrix} 0 \\ S_{22}^{-1} (S_{1n} - Q_{1n}) \end{bmatrix}.
\]

2) If \( S - Q \) is positive definite and if \( \det[S - Q] \neq \det[S] \), then \( 0 < a_n^2 < 1. \) The inverse solution of the Lyapunov equation yields two system matrices. The unknown elements of the system matrices are given by

\[
a_1 = \begin{bmatrix}
a_n \\ a_{n-1}
\end{bmatrix} = \begin{bmatrix} \sqrt{\det[S - Q]/\det[S]} \\ S_{22}^{-1} [(S_{1n} - Q_{1n}) - S_{12} \sqrt{\det[S - Q]/\det[S]}] \end{bmatrix}.
\]

\[
a_2 = \begin{bmatrix}
-\frac{a_n}{a_{n-1}} \\
\frac{\sqrt{\det[S - Q]/\det[S]}}{S_{22}^{-1} [(S_{1n} - Q_{1n}) + S_{12} \sqrt{\det[S - Q]/\det[S]}]}
\end{bmatrix}.
\]

3) If \( \det[S - Q] = \det[S] \), then \( a_n^2 = 1 \) and the inverse solution of the Lyapunov equation yields two unstable system matrices. All the eigenvalues of the unstable system matrices lie on the unit circle.

Theorem 3.2: The necessary conditions for the existence of stable solutions to the inverse solution of the Lyapunov equation are:

i) \( S > 0 \)

ii) \( Q \geq 0 \)

iii) \( S - Q > 0 \) or \( S - Q \geq 0 \)

iv) \( \det[S - Q] \neq \det[S] \).

Note that the above conditions are only necessary but not sufficient for the existence of stable solutions.

IV. THE GENERATION OF \( q \)-MARKOV COVERS

The above algorithm for obtaining the system matrix by solving the Lyapunov equation inversely can be easily used for generating \( q \)-Markov Covers. Before we present a method for the generation of \( q \)-Markov Covers, we give a method to compute the impulse response Gramian from the output covariances and the Markov parameters of the system.

Lemma 4.1: The impulse-response Gramian \( S \) of an \( n \)-th order system can be expressed as follows:

\[
S = S_n - M_p M_q^T
\]

(4.1)

where

\[
S_n = \begin{bmatrix}
\xi_{11} & \xi_{12} & \xi_{13} & \cdots & \xi_{1, n} \\
\xi_{12} & \xi_{11} & \xi_{12} & \cdots & \xi_{1, n-1} \\
\xi_{13} & \xi_{12} & \xi_{11} & \cdots & \xi_{1, n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\xi_{1, n} & \xi_{1, n-1} & \cdots & \xi_{12} & \xi_{11}
\end{bmatrix}
\]

and \( M_q \) is a lower triangular Toeplitz matrix of Markov parameters

\[
M_q = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & h_1 & 0 & \cdots & 0 \\
h_1 & h_1 & 0 & \cdots & 0 \\
h_2 & h_1 & h_1 & \cdots & 0 \\
h_3 & h_2 & h_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
h_{n-1} & h_{n-2} & h_{n-3} & \cdots & h_1
\end{bmatrix}
\]

with \( \xi_{1i} = R_i, i = 1, 2, \cdots, n \), the output covariances of the system. The \( h_i, i = 1, 2, \cdots, n - 1 \) are the first \( n - 1 \) Markov parameters of the system.
If we are given the impulse-response data (n covariances and Markov parameters) of an unknown system, the impulse-response Gramian S and matrix Q can be easily computed. If the matrices S and Q satisfy the conditions of Theorem 3.1 then the system matrices \((A_1, A_2)\) can be easily obtained from the inverse solution of the Lyapunov equation (2.3) as shown in the preceding section. The input and output vectors of the unknown system follow immediately from the fact that the system is in controllability canonical form (see Theorem 2.1). This yields

\[
\mathbf{b} = [1 \ 0 \ \cdots \ 0]^T
\]

and

\[
\mathbf{c} = [h_1 \ h_2 \ \cdots \ h_n].
\]

Since the system realizations are in controllability canonical form, it follows immediately that \(\{A_1, b\}\) and \(\{A_2, b\}\) are always controllable.

**Lemma 4.2:** The realizations \(\{A_1, b, c\}\) and \(\{A_2, b, c\}\) obtained are q-Markov Covers of nth degree if and only if \(\{A_1, c\}\) and \(\{A_2, c\}\) are observable.

**Proof:** The proof of lemma follows from the fact that if \(\{A_1, c\}\) and \(\{A_2, c\}\) are, respectively, observable then stable system matrices \(A_1\) and \(A_2\) satisfy the Lyapunov equation with \(S > 0\) and \(Q = c^T c \geq 0\).

**Remark:** Note also that the stability of \(A_1\) and \(A_2\) implies observability of the pairs \(\{A_1, c\}\) and \(\{A_2, c\}\). Furthermore, since \(\{A_1, b\}\) and \(\{A_2, b\}\) are always controllable the stability of \(A_1\) and \(A_2\) implies minimality of \(\{A_1, b, c\}\) and \(\{A_2, b, c\}\).

**Lemma 4.3:** If \(\{A_1, b, c\}\) and \(\{A_2, b, c\}\) are q-Markov covers of degree n then

\[
\prod_{i=1}^{n} \lambda_i[A_1] = \prod_{i=1}^{n} \lambda_i[A_2]
\]

where \(\lambda_i[A]\) are the eigenvalues of the system matrices \(A_1\) and \(A_2\), respectively.

The proof of the lemma follows immediately from (3.7).

The above discussion can now be used to formulate an algorithm for the generation of q-Markov covers for discrete-time systems.

**Algorithm:** Given the Markov parameters \(h_i, i = 1, 2, \cdots, n\) and output covariances \(R_i, i = 1, 2, \cdots, n\), we can construct q-Markov covers of degree n as follows:

1. Compute the impulse-response Gramian using the Lemma 4.1 and the matrix Q using (2.3) and (4.3).
2. Check whether S and Q matrices satisfy Theorem 3.1. If conditions (i)-(v) are not satisfied, no solution exists.
3. Solve the Lyapunov equation and find the two system matrices \((A_1, A_2)\) using (3.8) and (3.9).
4. The input and output vectors \(b\) and \(c\) follow immediately from (4.2) and (4.3).
5. Check for the observability of the pairs \(\{A_1, c\}\) and \(\{A_2, c\}\). The realizations \(\{A_1, b, c\}\) and \(\{A_2, b, c\}\) are q-Markov Covers of nth degree if \(\{A_1, c\}\) and \(\{A_2, c\}\) are, respectively, observable.

Note that the above algorithm is conceptually simple because the q-Markov Covers are obtained using simple formulas for system matrices (3.8) and (3.9) and input and output matrices (4.2) and (4.3). On the other hand, the method of [2] is complicated and requires a number of steps for computing the q-Markov Covers. The proposed algorithm is also computationally efficient compared to the algorithm of [2]. The proposed algorithm requires only one matrix inversion and evaluation of two determinants. On the other hand the algorithm of [2] requires the evaluation of two matrix inversions, two matrix square roots and \(n + 2\) left inverses for obtaining q-Markov Covers of nth degree.

It is clear that the proposed technique can be used for identification, as well as for model reduction. If instead of \(n \times n\) matrices \(S\) and \(Q\), submatrices of \(S\) and \(Q\) of order \(r \times r\) are used (where \(r < n\)), then a reduced-order model of the original system will be obtained. It can be easily shown that this model matches the first \(r\) Markov parameters and the first \(r\) output covariances of the original system. Hence, it is a q-Markov cover of degree \(r\).

**V. EXAMPLE**

Find the class of all third-order discrete-time systems having Markov parameters

\[
\{h_1, h_2, h_3\} = \{3, \frac{1}{2}, \frac{9}{16}\}
\]

and covariances

\[
\{R_1, R_2, R_3\} = \{9.3968, 0.9016, 1.7683\}.
\]

The matrix Q in (2.3) obtained from the Markov parameters is given by

\[
Q = \begin{bmatrix}
9.0000 & 0.7500 & 1.6875 \\
0.7500 & 0.0625 & 0.1406 \\
1.6875 & 0.1406 & 0.3164
\end{bmatrix}.
\]

The impulse-response Gramian computed from the above data and (4.1) is given by

\[
S = \begin{bmatrix}
9.3968 & 0.9016 & 1.7683 \\
0.9016 & 0.3968 & 0.1516 \\
1.7683 & 0.1516 & 0.3343
\end{bmatrix}.
\]

The two q-Markov covers obtained by the proposed algorithm are given by \(\{A_1, b, c\}\) and \(\{A_2, b, c\}\) where

\[
A_1 = \begin{bmatrix}
0 & 0 & 0.0625 \\
1 & 0 & 0.2120 \\
0 & 1 & -0.3939
\end{bmatrix},
A_2 = \begin{bmatrix}
0 & 0 & -0.0625 \\
1 & 0 & 0.2500 \\
0 & 1 & 0.2500
\end{bmatrix},
\]

\[
\mathbf{b} = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 3.0000 \ 0.2500 \ 0.5625 \end{bmatrix}.
\]

**VI. CONCLUSION**

A new method for generating q-Markov covers for SISO discrete-time system has been proposed. It has been shown that given the first \(n\) Markov parameters and covariances, at most two q-Markov covers of degree \(n\) can be obtained by solving the Lyapunov equation inversely to get the system matrices. It is shown that the q-Markov Covers exist if and only if stable inverse solution to the Lyapunov equation exists. Furthermore, it is shown that if q-Markov Covers exist, they are controllable and observable. It is also shown that the absolute value of the product of eigenvalues of the q-Markov covers are equal.

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**REFERENCES**


A. Basics

Iterative path formulated optimal routing algorithm. In particular, it is demonstrated excellent convergence rate properties through extensive analysis. The number of nodes in the underlying graph increases. The analysis is motivated by a particular path formulated gradient projection algorithm. The performance depends on some quantitative measure. The types of performance measures employed by most optimal routing formulations, estimate, in some sense, the average delay associated with sending a packet of data to a typical destination node.

Abstract-A problem of optimal routing is a set of routes that yields the "best" network performance. Roughly speaking, an optimal routing is a set of routes that yields the "best" network performance. The goal in the present note is to determine the amount of time required for a class of iterative path formulated optimal routing algorithms to converge. The time complexity of a routing algorithm values, link capacity values, and the number of network nodes affect the time required for convergence. In order to achieve meaningful values, link capacity values, and the number of network nodes affect the time required for convergence.

The following notation is needed in order to formally state the hypothesis. Perhaps the simplest queuing model is the so-called M/M/1 model. In this model, the arrival rate is constant and the service rate is constant. The service rate is denoted by \( \lambda \) and the arrival rate is denoted by \( \mu \). The following notation is needed in order to formally state the hypothesis:

- \( W \): The set of OD pair requesting communication.
- \( P_w \): A generic OD pair in \( W \).
- \( x_p \): The flow rate on the path \( p \).
- \( C_{ij} \): The capacity of link \( (i, j) \).
- \( F_{ij} \): The service rate on link \( (i, j) \).
- \( D(F) \): The delay for traffic \( F \).
- \( W \): All links or nodes in the network.

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