On the Stability Robustness of Additively Perturbed Interconnected Systems

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Abstract—The concept of structured singular value is used to analyse the stability robustness of additively perturbed interconnected systems.

I. INTRODUCTION

The problem of robust stabilization has been studied by many authors [1,7,8,10]. Denote by \( P_0(s) \) the nominal plant, by \( \Delta(s) \) the unstructured additive perturbation and by \( C(s) \) the controller stabilizing \( P_0(s) \). Then the perturbed system will remain stable if and only if \( \sigma[\Delta(j\omega)] < \sigma^{-1}[G(j\omega)] \), where \( G(s) = C(I+P_0C)^{-1} \), \( \sigma[.] \) is the maximum singular value of the matrix \([.]\). However, this result may be unnecessarily conservative if information about \( \Delta(s) \) is available. To reduce the conservatism, various perturbation weightings were introduced [2]. In [6,9], the Perron-Frobenius theory of nonnegative matrices was invoked to find some bounds of the stability robustness for interconnected systems containing a number of subsystems which are weakly coupled. In [3], the \( \mu \) function (structured singular value) was introduced for nonconservative robustness analysis and stabilizer design in case of structured (diagonal) plant uncertainties.

The purpose of this paper is to consider the stability robustness of additively perturbed interconnected systems using the small \( \mu \) methodology. The system of concern is shown in Fig. 1. In this diagram the transfer function matrix \( P_0(s) \) is partitioned into subsystems and interconnections. The additive uncertainty \( \Delta(s) \) may perturb all the blocks of \( P_0(s) \), and the norm of any block in \( \Delta(s) \) is bounded by \( \delta[\Delta_{ij}(j\omega)] \leq \|\delta(j\omega)\| \) for some \( \mathcal{R}\mathcal{H}^\infty \) function \( \delta(s) \) with minimum phase. To use the small \( \mu \) methodology, the system is first transformed into an equivalent form shown in Fig. 2, where \( \Delta(s) \) is of block diagonal structure. Then, a necessary and sufficient condition for robust stability will be established in terms of the following singular value inequality:

\[
|\delta(j\omega)| < \mu^{-1}(H)
\]

where \( H(s) \) is a transfer function matrix which can be easily constructed from \( G(s) \). Some computation aspect of \( \mu(H) \) will be considered. Since no weak coupling between the subsystems is presupposed, the new result can be applied to a wider range of interconnected systems.

This paper is organized as follows. Section 2 contains some preliminary results of robust stability and small \( \mu \) theory. In Section 3 we develop the main results of robust stability for interconnected systems. An example will be then given in section 4. The main contribution of this paper and some problems, yet to be studied, are summarized in section 5.

On nomenclatures: for any square complex matrix \( M \), we denote by \( \sigma[M] \) its maximum singular value, by \( M^* \) its complex conjugate transpose. Given any complex vector \( z \), \( z^* \) indicates its complex conjugate transpose and \( \|\cdot\| \) its Euclidean norm. For any positive integer \( k \), we denote by \( 0_k \) the \( k \times k \) zero matrix, by \( I_k \) the \( k \times k \) identity matrix, by \( C_k \) the set of all \( k \) complex vectors and by \( C_{kxk} \) the set of all \( k \times k \) complex matrices. We call block-structure of size \( n \) any \( n \)-tuple \( \mathcal{K} = (k_1, k_2, \ldots, k_n) \) of positive numbers. For any positive scalar \( \delta \) (possibly \( \infty \)), we denote by \( \mathcal{X}_\delta \) the family of \( k \times k \) \( (k = \sum_{i=1}^n k_i) \) block diagonal matrices

\[
\mathcal{X}_\delta = \{\text{blockdiag}(\Delta_1, \Delta_2, \ldots, \Delta_n) : \Delta_i \in C^{k_i \times k_i}, \text{s.t. } \sigma[\Delta_i] \leq \delta\}
\]

and by \( \mathcal{U} \) the family of block unitary matrices

\[
\mathcal{U} = \{\text{blockdiag}(U_1, U_2, \ldots, U_n) : U_i \text{ is a } k_i \times k_i \text{ unitary matrix}\}
\]
II. PRELIMINARIES

We consider the system in Fig. 1.

![System Diagram]

Figure 1: The system of concern

Here $P(s) = P_0(s) + \Delta(s)$ is the plant to be controlled, $C(s)$ is the controller to be designed. $P(s)$ has a nominal value at $P_0(s)$. It is assumed that $P(s)$ and $P_0(s)$ have the same number of unstable poles, and the uncertainty $\Delta(s) \in \mathcal{R}L^\infty$. Denote by $A(P_0, \Delta)$ the class of all such plants. This class is said to be robustly stabilizable (by a single controller) if there exists a $C(s)$ which stabilizes all plants $P(s) \in A(P_0, \Delta)$. The following result is well known.

**Theorem 1:** [1] The class $A(P_0, \Delta)$ can be robustly stabilized by a single controller $C(s)$ if and only if

(a) $C(s)$ stabilizes the nominal plant $P_0(s)$, and
(b) $\det(I + P_0C + \epsilon \Delta C) \neq 0$ for all $\epsilon \in [0, 1]$ and $s \in \mathcal{D}$, where $\mathcal{D}$ is the Nyquist contour.

For unstructured uncertainty $\Delta(s)$ and under condition (a), condition (b) is true if and only if

$$\tilde{\sigma}[\Delta] < \sigma^{-1}[G]$$

where $G(s) = C(I + P_0C)^{-1}$.

In the following section, we will find some robust stability condition for interconnected systems. The results rely heavily on the concept of structured singular value. For this reason, its definition and some established results on this topic will be briefly presented in the following.

**Definition 1:** [2] The structured singular value $\mu(H)$ of a $k \times k$ matrix $H$ with respect to block-structure $\mathcal{K}$ is the positive number $\mu$ having the following property that

$$\det(I + H \Delta_d) \neq 0 \quad \forall \Delta_d \in \mathcal{X}_d$$

if and only if

$$\delta \mu(H) < 1.$$
follows:

\[
P_0(s) = \begin{pmatrix}
P_{11}(s) & P_{12}(s) & \ldots & P_{1q}(s) \\
P_{21}(s) & P_{22}(s) & \ldots & P_{2q}(s) \\
\vdots & \vdots & \ddots & \vdots \\
P_{q1}(s) & P_{q2}(s) & \ldots & P_{qq}(s)
\end{pmatrix}
\]  

(7)

\[
\Delta(s) = \begin{pmatrix}
\Delta_{11}(s) & \Delta_{12}(s) & \ldots & \Delta_{1q}(s) \\
\Delta_{21}(s) & \Delta_{22}(s) & \ldots & \Delta_{2q}(s) \\
\vdots & \vdots & \ddots & \vdots \\
\Delta_{q1}(s) & \Delta_{q2}(s) & \ldots & \Delta_{qq}(s)
\end{pmatrix}
\]  

(8)

Here \( P_{ii}(s) \) and \( \Delta_{ii}(s) \) \( \in (\mathbb{R}(s))^{p_i \times m_i} \) for \( i = 1,2,\ldots,q \) with \( \sum_{i=1}^{q} m_i = m \) and \( \sum_{i=1}^{q} p_i = p \).

In order to establish a robust stability test using the \( \mu \)-concept, we will first transform \( A(s) \) into a block diagonal form. To make use of the result presented in the previous section, we assume that \( \Delta_{ii}(s) \) is square. Then \( p_i = p_j = m_i = m_j = m \) for all \( i = 1,2,\ldots,q \). Further, we assume that the maximum singular value of \( A_{ij}(s) \) is bounded above by \( \| A_{ij} \|_2 \leq \delta |j\omega| \) for some \( \mathcal{KH}^\infty \) function \( \delta(s) \) with minimum phase.

The system in Fig. 1 can be represented by an equivalent form shown in Fig. 2, where

\[
\Delta_d(s) = \text{blockdiag} \left( \Delta_1^1(s), \Delta_2^2(s), \ldots, \Delta_q^q(s) \right)
\]

with

\[
\Delta_i^j(s) = \text{blockdiag} \left( \Delta_{ii}^1(s), \Delta_{ii}^2(s), \ldots, \Delta_{ij}^q(s) \right)
\]

and

\[
M(s) = \begin{pmatrix}
M_{11}(s) & M_{12}(s) \\
M_{21}(s) & M_{22}(s)
\end{pmatrix}
\]  

(9)

The transfer function matrix \( M(s) \) can be determined as follows. Denote

\[
F_1(M, \Delta_d) := M_{11} + M_{12} \Delta_d(I - M_{22} \Delta_d)^{-1} M_{21}.
\]

(10)

Then, being the transfer function matrix from \( v \) to \( y \), \( F_1(M, \Delta_d) \) must be \( P_0(s) + \Delta(s) \), i.e.

\[
M_{11} + M_{12} \Delta_d(I - M_{22} \Delta_d)^{-1} M_{21} = P_0(s) + \Delta(s)
\]

(11)

This equality holds for

\[
M_{11}(s) = P_0(s)
\]

\[
M_{12}(s) = \text{blockdiag} (Y_m, Y_m, \ldots, Y_m)
\]

\[
M_{21}(s) = \left( \begin{array}{c}
Z_m \\
Z_m \\
\vdots \\
Z_m
\end{array} \right)
\]

\[
M_{22}(s) = 0
\]

where

\[
Y_m = (I_\hat{m}, I_\hat{m}, \ldots, I_\hat{m})
\]

\[
Z_m = \text{blockdiag} (I_{\hat{m}}, I_{\hat{m}}, \ldots, I_{\hat{m})}
\]

(12)

(13)

From Theorem 1, the controller stabilizes the class \( \mathcal{A}(P_0, \Delta) \) if and only if

(a) \( C(s) \) stabilizes the nominal plant \( P_0(s) \), and

(b) \( \det(I + P_0 C + \epsilon \Delta C) \neq 0 \) for all \( \epsilon \in [0,1] \) and \( s \in \mathcal{D} \).

Throughout the rest of this paper, we assume that \( \mu \)-condition (a) is true. Then condition (b) is equivalent to \( \det(I + c \Delta C(I + P_0 C)^{-1}) \neq 0 \). Substituting \( \Delta = M_{12} \Delta_d M_{21} \) into (b) and using the identity \( \det(I + XY)^{-1} = \det(I + YX) \), we get

\[
\det(I + c M_{12} \Delta_d M_{21} C(I + P_0 C)^{-1}) = \det(I + c \Delta_d H)
\]

where

\[
H(s) := M_{21} C(I + P_0 C)^{-1} M_{12}
\]

\[
= M_{21} G(s) M_{12}.
\]

Corollary 1: The perturbed system with the controller \( C(s) \) remains stable if and only if \( \det(I + c \Delta_d H) \neq 0 \) for all \( \epsilon \in [0,1] \) and \( s \in \mathcal{D} \). Or, equivalently,

\[
\mu(H) \delta \| \Delta_{ij} \| < 1 \quad \forall \epsilon \in [0,1] \text{ and } i, j = 1,2,\ldots,q.
\]

Denote by \( Z \) the set of all \( q \cdot m \) complex vectors such that \( \| z \|^2 = 1/q \). Partition \( z \in Z \) and \( G(s) \) as

\[
z = \begin{pmatrix}
z_1 \\
z_2 \\
\vdots \\
z_q
\end{pmatrix}
\]

\[
G(s) = \begin{pmatrix}
G_1 \\
G_2 \\
\vdots \\
G_q
\end{pmatrix}
\]
with $z_l \in \mathbb{C}^n$ for $l = 1, 2, \ldots, q$. Let $Y_z$ be the family of all $q \cdot n$ complex vectors of the form:

$$y(z) = \sum_{l=1}^{q} \begin{bmatrix} U_l \\ U_{l+1} \\ \vdots \\ U_{(q-1)\ell+1} \end{bmatrix} z_l$$

where $U_l$ is an arbitrary unitary matrix for $i = 1, 2, \ldots, q$, $z \in \mathbb{C}^n$. Then

**Theorem 4:** For a given $s \in D$,

$$\mu^2(H) = \max \left\{ \alpha : \alpha \|z_l\|^2 = \|G_l y_l(z)\|^2, \quad l = 1, 2, \ldots, q, \quad y_l(z) \in Y_z \right\}.$$ 

**Proof:** We compute now the structured singular value of the matrix $H(s)$ for a given $s \in D$. Define the projection matrices as in Theorem 2, $A_k(\alpha)$ and the associated $W(\alpha)$, $x \in \mathbb{C}^n$ and $v \in \mathbb{R}^q$ as in (5) and (6) for $k = 1, 2, \ldots, q^2$, partition $x$ as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{q^2} \end{bmatrix}$$

where $x_i \in \mathbb{C}^n$, and rewrite the first $q^2$ nature numbers $k = 1, 2, \ldots, q^2$ as

$$k = q \cdot r + l$$

where $r = 0, 1, \ldots, q - 1$, $l = 1, 2, \ldots, q$. Then routine matrix manipulations show

$$\tilde{y}(x) := M_{12} x$$

and

$$M_{21}^T S_k M_{21} = \tilde{S}_k$$

where $\tilde{S}_k$ is the $q^2 \times q^2$ projection matrix

$$\tilde{S}_k = \text{blockdiag}(0_{a_1}, \ldots, 0_{a_t}, I_{a_1}, 0_{a_1}, \ldots, 0_{a_t}).$$

Note that $\tilde{S}_k = \tilde{S}_k^T \tilde{S}_k$. It is clear that

$$H^* S_k H = M_{12}^T G^* M_{21}^T S_k M_{21} G M_{12}$$

$$= M_{12}^T G^* \tilde{S}_k G M_{12}$$

$$= M_{12}^T G^* G_{12}$$

and

$$z^* A_k(\alpha) z = \alpha \|z_{q+l}\|^2 - \|G_l y_l(z)\|^2.$$

$$z^* A_k(\alpha) z = 0$$

if and only if

$$\alpha \|z_{q+l}\|^2 = \|G_l y_l(z)\|^2.$$

Hence, for any $l \in \{1, 2, \ldots, q\}$

$$\|z_l\| = \|z_{q+l}\| = \cdots = \|z_{((q-1)\ell+1)}\|$$

Denote by $y_l$ the $l$th summand in (16)

$$y_l = \begin{bmatrix} z_l \\ x_{q+l} \\ \vdots \\ x_{(q-1)\ell+1} \end{bmatrix}$$

for $l = 1, 2, \ldots, q$. Then from (20), we get

$$\|z_l\| = \|z_{q+l}\| = \cdots = \|z_{((q-1)\ell+1)}\|$$

since $\|z_l\| = 1$, and some unitary matrices $U_l$. Since $\|z_l\| = 1$, we have

$$\|z_l\|^2 + \|z_{q+l}\|^2 + \cdots + \|z_{((q-1)\ell+1)}\|^2 = \frac{1}{q}.$$

Hence, $y(x) := \tilde{y}(x) \in Y_z$. On noting that

$$\mu^2(H) = \max \{ \alpha : (19) \text{ is satisfied} \}$$

we get

$$\mu^2(H) = \max \{ \alpha : \alpha \|z_l\|^2 = \|G_l y_l(z)\|^2, \quad l = 1, 2, \ldots, q, \quad y_l(z) \in Y_z \}$$

**IV. An Example**

Suppose that the uncertainty $\Delta(s)$ has identical blocks, i.e. $\Delta_{ij}(s) = \Delta(s)$ for all $i, j = 1, 2, \ldots, q$. In this case, the structured singular value of $H(s)$ has an explicit solution. Partition $G(s)$ as

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1q}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2q}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{q1}(s) & G_{q2}(s) & \cdots & G_{qq}(s) \end{bmatrix}$$

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**Theorem 5:** \( \mu(H) = \bar{\sigma} \left[ \sum_{i,j=1}^{q} G_{ij} \right] \).

**Proof:** We show the following equivalence:

\[
\det(I + \Delta_e H) \neq 0 \iff \bar{\sigma} \left[ \tilde{\Delta}(j\omega) \right] < \bar{\sigma}^{-1} \left[ \sum_{i,j=1}^{q} G_{ij} \right].
\]

Let us define the \( m \times m \) matrices

\[
J := \begin{pmatrix}
I_m & I_m & \cdots & I_m \\
I_m & I_m & \cdots & I_m \\
\vdots & \vdots & \ddots & \vdots \\
I_m & I_m & \cdots & I_m
\end{pmatrix},
\]

\[
V := \begin{pmatrix}
I_m & 0 & \cdots & 0 \\
I_m & I_m & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
I_m & 0 & \cdots & I_m
\end{pmatrix}.
\]

Then

\[
\Delta(s) = \text{blockdiag} \left( \tilde{\Delta}, \tilde{\Delta}, \ldots, \tilde{\Delta} \right) \cdot J
\]

\[
V^{-1} J G V = \begin{pmatrix}
\sum_{k=1}^{q} G_{k1} & \sum_{k=1}^{q} G_{k2} & \cdots & \sum_{k=1}^{q} G_{kq} \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]

From

\[
\det(I + \Delta_e H) = \det(I + \Delta) = \det \left( I + c \tilde{\Delta} \cdot V^{-1} J GV \right)
\]

\[
= \det \begin{pmatrix}
I + \tilde{\Delta} \sum_{i,j=1}^{q} G_{ij} & \tilde{\Delta} \sum_{k=1}^{q} G_{k2} & \cdots & \tilde{\Delta} \sum_{k=1}^{q} G_{kq} \\
0 & I & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I
\end{pmatrix}
\]

we see that \( \det(I + \Delta_e H) \neq 0 \) if and only if

\[
\bar{\sigma} \left[ \tilde{\Delta}(j\omega) \right] < \bar{\sigma}^{-1} \left[ \sum_{i,j=1}^{q} G_{ij} \right].
\]

Now, we consider the system

\[
G(s) = \begin{pmatrix}
G_S(s) & G_I(s) \\
-G_I(s) & -G_S(s)
\end{pmatrix}
\]

Since \( \sum_{i,j=1}^{q} G_{ij}(s) = 0 \), the perturbed system is stable for all \( \tilde{\Delta}(s) \) such that \( \bar{\sigma} \left[ \tilde{\Delta}(j\omega) \right] < \infty \).

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**V. Conclusion**

In this paper it was shown that a nonconservative stability robustness test can be found for additively perturbed interconnected systems, using the small-\( \mu \) methodology. To complete this approach, the problem formulated in theorem 4 has to be solved.

**References**


