ALGEBRAIC NECESSARY AND SUFFICIENT CONDITIONS FOR THE STABILITY OF 2-D DISCRETE SYSTEMS

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ABSTRACT

Algebraic necessary and sufficient conditions for the stability analysis of 2-D discrete systems are presented. These conditions are developed based on the frequency dependent formulation of the Lyapunov equation using Kronecker products. It is shown that these necessary and sufficient conditions for internal stability of 2-D discrete systems are equivalent to testing the eigenvalues of constant matrices. This is a simplification over earlier tests which require testing the positivity of one or more functions of \( w \) for all \( w \in [0,2\pi] \).

1. INTRODUCTION

The interest in two-dimensional (2-D) systems is motivated by their numerous applications in many areas. Among the several problems associated with the design and implementation of such systems, stability is one which has been considered extensively in the literature during recent years [1]. In this context the extension of the Lyapunov equation, a standard tool for the stability analysis of 1-D systems, to the 2-D case is of considerable importance [2-6]. Such an extension leads to necessary and sufficient conditions for 2-D stability.

These conditions however require testing the positivity of one or more functions in \( w \) for all \( w \in [0,2\pi] \) and are of comparable complexity as other algebraic polynomial tests. Techniques to overcome these difficulties which require to test only a finite number of points for \( w \in [0,2\pi] \) instead of all points were proposed in [13] and [14].

2-D Stability conditions were also developed using a formulation of the Lyapunov equation using Kronecker products in [7, 8]. This approach was further pursued in [9] and [10] where the parametrized 1-D Lyapunov equation using Kronecker products was considered and new conditions for the nonexistence of zeros of the 2-D characteristic polynomial were developed. These conditions require testing the locations of eigenvalues of constant matrices and thus are simpler than the existing tests.

In this paper the frequency dependent Lyapunov equation is used to obtain simple necessary and sufficient algebraic stability tests. It will be shown that testing 2-D stability is equivalent to testing the location of the eigenvalues of constant matrices. The results presented here are an extension of the results presented in [10] (for \( \det A_{12} = 0 \)) and are illustrated with an example.

The tests presented here are simplification over existing stability tests which require to test the positivity of one or more function of \( w \) for all \( w \in [0,2\pi] \).

2. PRELIMINARIES

Linear shift invariant 2-D discrete systems can be represented by the following state-space model [15]:

\[
\begin{bmatrix}
    x^h(i+1,j) \\
    x^v(i,j+1)
\end{bmatrix} = \begin{bmatrix}
    A_{11} & A_{12} \\
    A_{21} & A_{22}
\end{bmatrix} \begin{bmatrix}
    x^h(i,j) \\
    x^v(i,j)
\end{bmatrix} + \begin{bmatrix}
    B_1 \\
    B_2
\end{bmatrix} u(i,j) \tag{1}
\]

\[
y(i,j) = \begin{bmatrix}
    C_1 \\
    C_2
\end{bmatrix} \begin{bmatrix}
    x^h(i,j) \\
    x^v(i,j)
\end{bmatrix} \tag{2}
\]

where \( x^h \in \mathbb{R}^n \) and \( x^v \in \mathbb{R}^m \) represent the horizontal and vertical states respectively, \( u \) is the input and \( y \) is the output. The system matrix is given by

\[
A = \begin{bmatrix}
    A_{11} & A_{12} \\
    A_{21} & A_{22}
\end{bmatrix} \tag{3}
\]

with \( A_{i,j}, i,j = 1,2 \) of appropriate dimensions. The stability of a 2-D system realized with a model of the above type depends on the zeros of the characteristic
polynomial \( C(z_1, z_2) \) given by
\[
C(z_1, z_2) = \det \begin{bmatrix} 1 - z_1 A_{11} & -z_1 A_{12} \\ -z_2 A_{21} & 1 - z_2 A_{22} \end{bmatrix}
\]  
(4)

The stability condition is given by \([1]:\)
\[
C(z_1, z_2) \neq 0 \text{ for } (z_1, z_2) \in \mathbb{D}^2
\]
(5)

The stability condition can be tested using the extension of the Lyapunov approach to the 2-D case. This has led to the following theorem:

**Theorem 1** [6]: The system (1) is internally stable iff
i) \( |\lambda_i[A_{22}]| < 1 \)
ii) the matrix equation
\[
H^T(e^{-j\omega})P(e^{j\omega}) - P(e^{j\omega}) = -Q(e^{j\omega})
\]
has a hermitian positive definite (h.p.d.) solution \( P(e^{j\omega}) \) for any given positive definite hermitian matrix \( Q(e^{j\omega}) \) for all \( \omega \in [0, 2\pi] \)

\[
H(e^{j\omega}) = A_{11} + A_{12}(Ie^{j\omega} - A_{22})^{-1} A_{21}
\]
(8)

Recently, the formulation of the Lyapunov equation using Kronecker products (denoted by \( \otimes \)) has been utilized in [7-10] to develop new stability conditions which are easy to test. Such tests are presented in the next section.

**3. A NEW STABILITY CONDITION**

In this section a new stability condition is presented and it is shown that it leads to a 2-D stability test which requires testing only the eigenvalues of constant matrices.

**Theorem 2** [10]: The system (1) is internally stable iff
i) \( |\lambda_i[A_{22}]| < 1 \)
ii) \( P \), the solution of
\[
H^T(e^{-j\omega})P(e^{j\omega}) - P(e^{j\omega}) = Q
\]
(10)

is positive definite for any given positive definite matrix \( Q \) and an arbitrary \( \omega_0 \in [0, 2\pi] \)

iii) \( \det(I - H^T(e^{-j\omega}) \otimes H^T(e^{j\omega})) \neq 0 \) for all \( \omega \in [0, 2\pi] \)

Theorem 2 requires testing the stability of two constant matrices, i.e., \( A_{12} \) and \( H(e^{j\omega}) \), and the testing of the determinant eq. (11) for all \( \omega \in [0, 2\pi] \). This is a similar condition like the one presented in [8]. Testing stability using the above theorem, however, is not simpler than the existing polynomial tests. It is required to evaluate the determinant in eq. (11) and test the existence of zeros of this polynomial for all \( \omega \in [0, 2\pi] \). Condition iii) of the above theorem can be modified to a simpler form as was shown in [10].

**Theorem 3** [10]: The system (1) is stable iff
i) \( |\lambda_i[A_{22}]| < 1 \)
ii) \( |\lambda_i[H(e^{j\omega})]| < 1 \) for an arbitrary \( \omega_0 \in [0, 2\pi] \)
iii) \( \det(\lambda^2 X_1 + \lambda X_2 + X_3) \neq 0 \) for \( |\lambda| = 1 \)

where
\[
X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_{22}^* \otimes I \end{bmatrix}
\]
(15)
\[
X_2 = \begin{bmatrix} 0 & A_{11}^* \otimes A_{11} & 0 & A_{11}^* \otimes A_{12}^* \\ -A_{12}^* \otimes I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ I - A_{12}^* \otimes A_{12} & 0 & I \otimes A_{22} & 0 \end{bmatrix}
\]
(16)
\[
X_3 = \begin{bmatrix} 0 & A_{11}^* \otimes A_{11} & 0 & A_{11}^* \otimes A_{12}^* \\ I - A_{12}^* \otimes A_{12} & 0 & I \otimes A_{22} & 0 \\ A_{11}^* \otimes A_{11} & 0 & -I \otimes A_{22} & 0 \\ A_{11}^* \otimes A_{12} & 0 & 0 & -I \otimes A_{22}^* \end{bmatrix}
\]
(17)

The proof of this theorem can be found in [10].

The above condition for stability requires the testing of the eigenvalues of two constant matrices and of a second order matrix polynomial. The difficulty with this condition is that matrix \( X_1 \) is a singular matrix and, therefore, the solutions of eq. (14) cannot be obtained directly using existing software packages. It was shown in [10] that if \( \det A_{22} \neq 0 \), then the stability condition can be further modified to lead to a condition which can be easily tested.

**Theorem 4** [10]: The system (1) with \( \det A_{22} \neq 0 \) is internally stable iff
i) \( |\lambda_i[A_{22}]| < 1 \)
ii) \( |\lambda_i[H(e^{j\omega})]| < 1 \) for an arbitrary \( \omega_0 \in [0, 2\pi] \)
iii) \( \det(\lambda^2 Y_1 + \lambda Y_2 + Y_3) \neq 0 \) for \( |\lambda| = 1 \)

where
\[
Y_0 = \begin{bmatrix} I & -(A_{12}^* \otimes I) Y_2 & 0 & 0 \\ 0 & -(I \otimes A_{22}^*) & -(A_{12}^* \otimes A_{22}^*) Y_2 & 0 \\ 0 & 0 & 0 & -I \\ 0 & (A_{12}^* \otimes I)^{-1}(A_{22}^* \otimes A_{22}^*) Y_3 & Y_3 & Y_4 \end{bmatrix}
\]
(22)
\[
Y_1 = \begin{bmatrix} -(A_{12}^* \otimes I) & -(A_{12}^* \otimes I) Y_6 & 0 & Y_6 \\ -(A_{12}^* \otimes A_{12}^*) Y_5 & I & Y_6 & 0 \\ 0 & 0 & I & 0 \\ (A_{12}^* \otimes I)^{-1}(A_{22}^* \otimes A_{22}^*) Y_6 & 0 & Y_8 & I \end{bmatrix}
\]
(23)

and
\[
Y_2 = (I - A_{12}^* \otimes A_{12}^* )^{-1}(I \otimes A_{22}^*)
\]
(24)
\[ Y_2 = (A_{22}^T \otimes I)^{-1}(I \otimes A_{22}^T) \]  
(25) 
\[ Y_4 = - (A_{22}^T \otimes I)^{-1}(I + A_{22}^T \otimes A_{22}^T) \]  
(26) 
\[ Y_5 = (I - A_{12}^T \otimes A_{22}^T)^{-1}(A_{12}^T \otimes A_{22}^T) \]  
(27) 
\[ Y_6 = (A_{12}^T \otimes I)^{-1}(A_{12}^T \otimes A_{22}^T)Y_7 \]  
(28) 
\[ Y_7 = (I - A_{11}^T \otimes A_{22}^T)^{-1}(A_{11}^T \otimes A_{22}^T) \]  
(29) 
\[ Y_8 = -(A_{22}^T \otimes I)Y_7 \]  
(30) 
\[ Y_9 = -(A_{12}^T \otimes A_{22}^T)Y_7 \]  
(31)

The proof of this theorem can be found in [10].

The above test can be easily tested using existing software packages. It requires testing the eigenvalues of three constant matrices and solving a generalized eigenvalue problem for which software is widely available.

Consider now the case \( \det A_{22} = 0 \). This leads to the following stability test.

**Theorem 5:** The system (1) with \( \det A_{22} = 0 \) is internally stable iff

i) \( |\lambda_1[A_{22}]| < 1 \)  
(32) 

ii) \( |\lambda_i[H(e^{j\omega_0})]| < 1 \) for an arbitrary \( \omega_0 \in [0, 2\pi] \)  
(33) 

iii) \( \det(\lambda W_1 + W_0) \neq 0 \) for \( |\lambda| = 1 \)  
(34) 

with

\[
W_1 = \begin{bmatrix}
K_{11} & K_{12} & 0 \\
0 & I & 0 \\
\Lambda^{-1}K_{21} & 0 & I
\end{bmatrix}
\]  
(35) 
\[
W_0 = \begin{bmatrix}
K_{10} & K_{30} & 0 \\
0 & 0 & -I \\
\Lambda^{-1}K_{20} & \Lambda^{-1}K_{40} & \Lambda^{-1}K_{41}
\end{bmatrix}
\]  
(36)

where \( A \) and \( T \) are obtained from

\[-T^{-1}(A_{22}^T \otimes I)T = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda \end{bmatrix} \]  
(37)

with

\[
\det(A) \neq 0 
\]  
(38) 
\[
\det(T) \neq 0 
\]  
(39)

and \( K_{10} \ldots K_{41} \) are obtained by properly partitioning the left hand side of the following equation.

\[
\begin{bmatrix}
\lambda^2X_1 + \lambda X_2 + X_3 \\
X_1 \\
0
\end{bmatrix}
\begin{bmatrix}
I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & T^{-1}
\end{bmatrix}
= 
\begin{bmatrix}
K_{11} + K_{10} & K_{21} + K_{10} & K_{31} + K_{30} & K_{41} + K_{40} \\
\end{bmatrix}
\]  
(40)

where \( X_1, X_2 \) and \( X_3 \) are given by eqs. (15), (16) and (17).

**Proof:** The first two conditions are identical with the first two conditions of theorem 3. It is therefore, necessary that condition iii) is equivalent to condition iii) of theorem 3. Consider eq. (37). There always exist \( A \) and \( T \) such that eqs. (37) to (39) are satisfied. One such possible choice is \( A \) being a matrix in Jordan form with all eigenvalues of \( -(A_{22}^T \otimes I) \) not equal to zero and \( T \) the matrix of all eigenvectors. With \( X_1, X_2 \) and \( X_3 \) given by eqs. (15)-(17), this implies that the left hand side of (40) can be always decomposed as indicated in the right hand side of (40). Condition iii) of theorem 3 is then equivalent with the determinant of the right hand side of (40) not having zeros for \( |\lambda| = 1 \). Using a similar approach as in the second half of the proof of the previous theorem it can be shown that this is equivalent with conditions iii) of theorem 5.

The stability tests outlined in theorems 4 and 5 can be easily extended to the cases \( \det A_{11} = 0 \) and \( \det A_{11} \neq 0 \) which leads to 4 tests. Obviously, in the case where \( \det A_{11} = \det A_{22} = 0 \) theorem 5 (or the equivalent for \( \det A_{11} = 0 \)) has to be used. In the case where one of \( \det A_{22} \) and \( \det A_{11} \) is zero and the other one is not, one can choose between different possible tests. A criterion which can be used for that choice is the size of matrices \( Y_1, Y_0 \) and \( W_1, W_0 \). It can be easily seen from theorems 4 and 5 that

- size of \( Y_0, Y_1 = 3m^2 + n^2 \)
- size of \( W_0, W_1 = 3m^2 + n^2 + n_A \)

where

\[
m = \text{size of } A_{22} \\
n = \text{size of } A_{11} \\
n_A = \text{rank}(A)
\]

The test which leads to matrices with the smallest possible size can be used. The proposed test will be illustrated with an example in the next section.

**4. EXAMPLE**

Consider the matrix

\[
A = \begin{bmatrix}
-0.5 & 0.75 & 0.3895 & 0.03895 \\
0 & 0 & 0 & 0 \\
0.1423 & 0 & -0.4 & 0.2 \\
-0.0342 & 0 & -0.3 & 0.3
\end{bmatrix}
\]

where \( n = m = 2 \). It can be easily seen that

\[
\det(A_{11}) = \det(A_{22}) = 0
\]

and, therefore, the stability test of theorem 5 has to be used. This leads to

\[
|\lambda_i[A_{22}]| = [0.0, 0.37]
\]

for the first condition

\[
|\lambda_i[H(e^{j\omega_0})]| = [0.0, 0.4694]
\]
for the second condition with $\omega_0 = 1$. For the 3rd condition the fact that
\[
\text{rank } (A_1^T \otimes I) = 2
\]
implies that
\[
L_{11}, L_{10} \text{ are } 14 \times 14 \text{ matrices} \\
L_{12}, L_{20} \text{ are } 14 \times 2 \text{ matrices} \\
L_{13}, L_{30} \text{ are } 2 \times 14 \text{ matrices} \\
A, L, 1, L, o \text{ are } 2 \times 2 \text{ matrices}
\]
and consequently
\[
Y_o, Y_1 \text{ are } 18 \times 18 \text{ matrices}
\]
The condition
\[
det(\lambda Y_1 + Y_o) = 0
\]
is satisfied for $\lambda$ with
\[
|\lambda| = [0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 17.0206, 2.3762, 2.3762, 2.0288, 0.3700, 0.2180, 0.1245, 0.3707, 0.4343, 2.7027, 0.0000, 0.0000, 0.0000]
\]
From theorem 5 follows immediately that the matrix $A$ has a stable 2-D characteristic polynomial.

5. CONCLUSIONS

Necessary and sufficient algebraic stability conditions for 2-D discrete systems have been presented. They are based on the frequency dependent formulation of the Lyapunov equation for 2-D discrete systems using the Kronecker product. The proposed stability test requires testing the eigenvalues of constant matrices only and thus is simpler than existing tests which require testing the positivity of one or more functions of $\omega$ for all $\omega \in [0, 2\pi]$. If can be further extended to test the very strict Hurwitz property of a 2-D characteristic polynomial [12].

REFERENCES