Single-channel controllability of interconnected systems

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Abstract: A system \((A, B_d)\) with two input channels which correspond to its two interconnected subsystems \((A_{11}, B_{11})\) and \((A_{22}, B_{22})\), is considered. The conditions for \((A, B_d)\) to be controllable by only one of the input channels are then formulated. The conditions found replace the usual controllability conditions and involve only certain submatrices of the initial system matrices \(A\) and \(B_d\); as such, they are particularly well suited for the analysis of large-scale systems.

1 Introduction and problem statement

A two input channel system \((A, B_d)\), defined in the state-space by an equation of the form \(\dot{x} = Ax + B_d u\), is considered. The matrices \(A\) and \(B_d\) are in the form

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B_d = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}
\]

and it is supposed that \((A_{11}, B_{11})\) is controllable. The conditions for which system \((A, B_d)\) will also be controllable must then be found, without building the full controllability matrix of \((A, B_d)\).

The above problem is related to the following general situation: let \((A, B)\) be a system defined by the equations, \(\dot{x} = A x + B u\), and \(\dot{x}_2 = A_{22} x_1 + A_{23} x_2 + B_{22} u\). These equations can be written compactly in the form \(\dot{x} = Ax + B u\), where \(x = [x_1 \ x_2]^{\top}\), \(u = [u_1 \ u_2]^{\top}\) and matrices \(A\) and \(B\) are as in eqn. 1. System \((A, B_d)\) is a two-input channel interconnected system ('global system'), of state-vector \(x\) and input \(u\). It consists of the two interconnected subsystems \((A_{i1}, B_{i1})\), \(i = 1, 2\), of state vectors \(x_1\) and \(x_2\). The vectors \(u_1\) and \(u_2\) correspond to the two input channels of the global system, and the terms \(A_{12} x_2\) and \(A_{22} x_1\) represent the interconnections between its two subsystems.

It is supposed that system \((A, B_d)\) is controllable, as well as each subsystem \((A_{i1}, B_{i1})\), \(i = 1, 2\). During the operation of this system, one of its input channels might fall out; in this case it is relevant to ask if the system would still be controllable with the remaining input channel. It is in this sense that this problem could be considered as a problem of the 'fault-tolerant linear control systems' [7]. The formalisation of this situation leads to the problem posed.

Another problem related to that above is the problem of implementing a local feedback (for example of the form \(u_2 = K_{22} x_2\)) around one of the input channels of the two-input channel interconnected system \((A, B_d)\), and then to investigate if the resulting closed-loop system is controllable with the other input channel; i.e. if \((A + B_2 K_{22}, B_1)\) is controllable, for some \(K_{22}\). Of course, here we are interested in the controllability of the open-loop system \((A, B_d)\), corresponding to a zero feedback matrix \(K_{22}\). This problem was initially investigated in References 1 and 2 and its solution was formulated in what Corfmat and Morse called 'complete systems'. Later, the solvability conditions were expressed in terms of the 'decentralised fixed modes' of \((A, B_d)\) [11], as well as of the 'transmission zeros' of various subsystems of \((A, B_d)\) [9]. We shall not pursue any further the connections of this problem to that investigated in the present paper.

It is well known that the properties of the controllability can be expressed in terms of the 'decoupling zeros' of the system [8]. The present paper adopts a pure state-space point of view, both for the definition of the problem and for the formulation of its solution, and certain methods, which were developed for systems defined in the frequency domain, are used for the derivation of the solution. It is shown that, for the controllability of the single-input channel system \((A, B_i)\), only the subsystem \((A_{22}, A_{21})\) (see below for an explanation of these symbols) is responsible; however, the exact relationship between the decoupling zeros of \((A, B_i)\) and those of \((A_{22}, A_{21})\) is unknown. Finally, it must be added that the problem of the single-channel controllability could also be defined in the frequency domain (i.e. for systems defined by their transfer functions); however, the solution of the problem formulated in this form remains, at this point, open.

2 Preliminaries

We suppose that both input channels of system \((A, B_d)\) are scalar; in this case the matrices \(B_{11}\) and \(B_{22}\) are column vectors written as \(b_{11}\) and \(b_{22}\).

We consider system \((A, B_i)\) as in eqn. 1. Without loss of generality, we may assume that submatrix \(A_{22}\) of \(A\) is cyclic. Let \(b_{22}\) be a vector such that \((A_{22}, b_{22})\) is a controllable pair. We also assume that \((A, B_i)\), with

\[
B_i = \begin{bmatrix} b_{11} \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}
\]

is controllable.

We suppose that each \(n_i\) dimensional controllable pair \((A_{ii}, b_{ii})\), \(i = 1, 2\), is in its companion controllable form,
with

\[ A_i = \begin{bmatrix} 0 & 1 \\ \vdots & \vdots \\ a_{ij} & 1 \end{bmatrix}, \quad b_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \]  

(3a)

The submatrices \( A_{ij} \) (\( i \neq j \)) of \( A \) then take the form

\[ A_{ij} = \begin{bmatrix} a_{ij} \\ \vdots \\ a_{nj} \end{bmatrix} \]  

(3b)

\( (a_{ij}, \text{as in eqn. 3}) \), denotes the transpose of the column vector \( a_{ip} \).

With \( (A, B) \) in this form, we define the \((n-2) \times n\) polynomial matrix \( D(s) \), termed the 'intercontrollability matrix' [3] of system \( (A, B) \), to be

\[ D(s) = \begin{bmatrix} s & -1 & \cdots & -A_{21}^0 & \cdots & -A_{n1}^0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -A_{2n}^0 & s & \cdots & -1 \end{bmatrix} \]  

(4)

with which the conditions for the controllability of \( (A, B) \) as in eqns. 1, 2 and 3, can be expressed.

In the development to follow, the determination of the kernel of the matrix \( D(s) \) of a controllable system \( (A, B) \) is crucial. To state this result the following is needed: Let \( D(s) \) be a controllable system as in eqns. 2 and 3, and \( (A, B) \) be a controllable pair as in eqns. 2 and 3. System \( (A, B) \) is then controllable if and only if the polynomials \( \{m_{21}(s), m_{22}(s)\} \) of \( M(s) \) as in eqn. 6 are relatively prime.

2.1 Theorem 1 [3]

Let \( (A, B) \) be a controllable system as in eqns. 2 and 3, and \( D(s) \) its intercontrollability matrix. The kernel \( U(s) \) of \( D(s) \) is then as follows:

(i) if rank \( \{G\} = 1 \), \( U(s) = \begin{bmatrix} TS_{n-2}(s) & 0 \\ -X(s) & \varepsilon_1 \\ 0 & \varepsilon_2 \end{bmatrix} \)

(ii) if rank \( \{G\} = 2 \), \( U(s) = \begin{bmatrix} TS(s) \\ -G^{-1} \delta(s) \end{bmatrix} \)

3 Main result

The controllability of the pair \( (A, B) \) is examined by investigating its transfer function \( (sI - A)^{-1}b_1 \). To achieve this, we assume that \( (A, B) \) is 'imbedded' in the interconnected system \( (A, B) \), with \( A \) as in eqn. 3 and \( B \), as in eqn. 2. The 'structure identity' [3] of this system has the form

\[ (sI - A)U(s) = B_s M(s) \]  

(5)

where \( U(s) \) is the kernel of \( D(s) \) and

\[ M(s) = \begin{bmatrix} m_{11}(s) & m_{12}(s) \\ m_{21}(s) & m_{22}(s) \end{bmatrix} \]

\[ A_s = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \]

and \( E = \text{diag} \{ e_{11}, e_{22} \} \)

the \( n_i \) dimensional vectors \( e_{ij}, i = 1, 2 \), being of the form \( e_{11} = [0 \cdots 0 1]^T \).

3.1 Theorem 2

Let \( (A, B) \) be a system as in eqn. 1. Suppose that \( b_2 \) is a vector such that the pair \( (A, B_2) \), with \( B_2 \), as in eqn. 2, is controllable and that it is as in eqn. 3. System \( (A, B) \) is then controllable if and only if the polynomials \( \{m_{21}(s), m_{22}(s)\} \) of \( M(s) \) as in eqn. 6 are relatively prime.

3.2 Proof

Using the structure identity, eqn. 5, the transfer matrix of \( (A, B) \) takes the form

\[ (sI - A)^{-1}b_1 = U(s) \begin{bmatrix} M_1(s) & {} \\ -M_2(s) & {} \end{bmatrix}_{1st\ column} \]

\[ = \begin{bmatrix} 1/\det M(s) & U(s) \end{bmatrix} \begin{bmatrix} m_{12}(s) \\ -m_{21}(s) \end{bmatrix} \]

with \( \det [M(s)] = m_{11}(s)m_{22}(s) - m_{12}(s)m_{21}(s) \). In case (i) of theorem 1

\[ (sI - A)^{-1}b_1 = \begin{bmatrix} 1/\det M(s) & \begin{bmatrix} TS_{n-2}(s) & 0 \\ -X(s) & \varepsilon_1 \\ 0 & \varepsilon_2 \end{bmatrix} \\ -G^{-1}\delta(s) & -m_{21}(s) \end{bmatrix} \]

while in case (ii) of theorem 1

\[ (sI - A)^{-1}b_1 = \begin{bmatrix} 1/\det M(s) & \begin{bmatrix} TS(s) \\ -G^{-1}\delta(s) \end{bmatrix} \\ -m_{22}(s) \end{bmatrix} \]

In both cases the transfer function \( (sI - A)^{-1}b_1 \) can be written in the equivalent form of a left matrix fraction description, \( X^{-1}(s)Y(s) \), with \( X(s) = \det M(s) \). It is known that system \( (A, B) \) will be controllable if and only if the polynomial matrices \( \{X(s), Y(s)\} \) are relatively left-prime [4], i.e. if and only if rank \( \{X(z), Y(z)\} = 1 \) for all complex numbers \( z \). The statement of the theorem follows by considering the specific forms of \( X(s) \) and \( Y(s) \).

QED

3.3 Interconnected systems of the CCM-type

An important special case of an interconnected system \( (A, B) \) is when it is of the component connection model type (CCM-type) [5]. In this case the canonical forms of \( A \) and \( B \), as used in the previous development, take a particularly simple form as \( (A, B) \) is already in its multi-component controllable form, with, in eqn. 3b, \( A_1^0 = 0 \) and \( A_2^0 = 0 \). The intercontrollability matrix of this type of system is then also particularly simple, as it is equal to

\[ D(s) = \begin{bmatrix} s & -1 \\ \vdots & \vdots \\ s & -1 \end{bmatrix} \]

Its kernel \( U(s) \) is as given in case (ii) of theorem 1 with \( T = I_{n-2}, G_a = -I_2 \) and \( \delta(s) = \text{diag} \{ e_{11}, e_{22} \} \), with \( d_i = n_i - 1 \) for \( i = 1, 2 \). Matrix \( M(s) \), as in eqn. 6, is actually the
characteristic polynomial matrix of \( A \)

\[
M(s) = \text{diag} \{ s^n, s^m \} - A_m \mathbf{S}(s) = \begin{bmatrix} X_{11}(s) & a_{12}(s) \\ a_{21}(s) & X_{22}(s) \end{bmatrix}
\]

with

\[
\mathbf{S}(s) = \text{diag} \{ S_{11}(s), S_{22}(s) \}
\]

\[
S_m(s) = \begin{bmatrix} 1 & s & \cdots & s^{n-1} \end{bmatrix}
\]

In this case, \((A, b, b_1)\) will be controllable if and only if the polynomials \([a_{12}(s)\) and \(X_{22}(s)\)] are prime. This is a result that appeared initially in Reference 6. It is remarked that the analysis of the single-channel controllability is particularly simple for this class of systems because it involves the relative primeness of the scalar polynomials \([a_{12}(s)\) and \(X_{22}(s)\), which are very simple to determine. Additionally, in the case of a sudden change of the values of certain interconnection parameters of the system, as might be expressed by the change of \(A_{21}\) into \(A_{21}\), resulting in a change of the polynomial \(a_{12}(s)\) into the new polynomial \(\tilde{a}_{12}(s)\), it still involves the relative primeness of the new polynomial \(\tilde{a}_{12}(s)\) and of the old one \(X_{22}(s)\). This property makes the analysis particularly attractive for large-scale systems, especially when they are repeated and when sudden changes in the interconnection parameters occur.

### 3.4 Example

We consider a system \((A, b_1)\) with

\[
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad b_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}
\]

subsystem \((A, b, b_1)\) is as indicated by the partitioning of \(A\) and \(b_1\). An obvious choice for \(b_{22}\) is \(b_{22} = [0 \ 1]^T\), for which the pair \((A, B_2)\), with \(B_2 = \text{diag} [b_{11}, b_{22}]\), is controllable. The intercontrollability matrix of this system is

\[
D(s) = \begin{bmatrix} s^3 - 1 & 0 & -1 \\ -2 & 1 & s \end{bmatrix}
\]

The permuted \(D(s)\) matrix is

\[
\tilde{D}(s) = D(s)P = \begin{bmatrix} s & 0 & -1 & -1 \\ -2 & s & -1 & 1 \end{bmatrix} = [sI - F] G
\]

with

\[
P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\]

\[
F = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \quad G = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}
\]

It follows that \(\text{rank} \ [G] = 2\), implying that \(U(s)\), as is in case (ii) of theorem 1. The matrix transforming \((\tilde{F}, G)\) into its multicompanion controllable form \((\tilde{F}, G)\) is

\[
\mathbf{T} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}
\]

with

\[
\tilde{F} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \tilde{G}_s
\]

The controllability indices of \((\tilde{F}, G)\) are \(d_1 = d_2 = 1\) and

\[
\delta(s) = [s - 1 \ 1] [s - 1 \ 1]
\]

4 Conclusions

A necessary and sufficient condition for the controllability of an interconnected system \((A, B_0)\), with only one of its two input channels, was given. Its derivation relied on the intercontrollability matrix \(D(s)\) of \((A, B_0)\) and the analytic determination of its kernel \(U(s)\). This permitted us to study the initial problem, posed in the state-space, with methods for systems defined with a matrix-fraction description. The conditions found, replace the usual controllability conditions, which involve the full matrices \(A\) and \(B_0\); they now involve only certain submatrices of \(A\) and are particularly well-suited for the controllability analyses of large-scale systems.

The case of the component connection model was treated in detail, and a numerical example was also included. Two open problems (in Sections 1 and 3) were also indicated.

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