Robust Stability of Interval Polynomials and Kharitonov’s Theorem

In 1978 V. L. Kharitonov published a remarkable paper on the stability of linear continuous dynamic systems whose characteristic equations have coefficients each lying between given limits (called interval polynomials). He gave necessary and sufficient conditions for all zeros of this family of polynomials to have negative real parts. Since then his results have been used by several researchers in different directions and some of these areas are overviewed in this article.

1. Kharitonov Theorem and Closely Related Results

Consider the family of polynomials

\[ f(s) = s^n + a_1 s^{n-1} + \cdots + a_n \]  

(1)
Robust Stability of Interval Polynomials and Kharitonov's Theorem

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1. Kharitonov Theorem and Closely Related Results

Consider the family of polynomials

\[ f(s) = s^n + a_1 s^{n-1} + \cdots + a_n \]  

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Robust Stability of Interval Polynomials and Khariitonov's Theorem

where

$$a_i \in [\tilde{a}_i, \bar{a}_i], \quad \tilde{a}_i \leq a_i \leq \bar{a}_i, \quad i = 1, 2, \ldots, n \quad (2)$$

Find conditions under which all zeros of all polynomials of the type of Eqn. (1) given Eqn. (2) have negative real parts.

**Theorem 1 (weak Khariitonov theorem).** A necessary and sufficient condition that all polynomials of the type of Eqn. (1) given Eqn. (2) have negative real parts (Hurwitz polynomials) is that the family of the 2^n polynomials in which each coefficient a_i is equal to either \(a_i\) or \(\bar{a}_i\) (i.e., the corners of the rectangular box in the n-dimensional coefficient space) are Hurwitz.

The proof of this theorem is based on the Hermite–Bieler theorem which implies that, in order that \(f(s) = h(s^0) + s g(s^0)\) be Hurwitz, it is necessary and sufficient that \(h(\lambda)\) and \(g(\lambda)\) have zeros which must be distinct, real, negative, and interlacing.

**Theorem 2 (strong Khariitonov theorem).** In order that the family of polynomials of the type of Eqn. (1) given Eqn. (2) are Hurwitz, it is necessary and sufficient that the following four polynomials are Hurwitz (only 4 corners of the rectangular box):

$$f_1(s) = \bar{a}_n + \sum_{i=1}^{n} \tilde{a}_i s^i + \sum_{i=1}^{n-1} \bar{a}_i s^i$$

$$f_2(s) = \bar{a}_n + \sum_{i=1}^{n} \bar{a}_i s^i + \sum_{i=1}^{n-1} \tilde{a}_i s^i$$

$$f_3(s) = \bar{a}_n + \sum_{i=1}^{n} \tilde{a}_i s^i + \sum_{i=1}^{n-1} \bar{a}_i s^i$$

$$f_4(s) = \bar{a}_n + \sum_{i=1}^{n} \bar{a}_i s^i + \sum_{i=1}^{n-1} \tilde{a}_i s^i$$

(3)

The proof of this theorem is also based on the H Hermite–Bieler theorem. In Bose (1985) a network theoretical proof is given which is useful for generalizations to multidimensional systems. This theorem is also valid for non monic polynomials.

### 1.1 Robust Hurwitz Low Order Polynomials

In Anderson et al. (1987) it is proved that for \(n = 3\) the Hurwitz property of \(f_1(s)\) is necessary and sufficient for the stability of the whole family. For \(n = 4\) only \(f_1(s)\) and \(f_2(s)\) are needed, and for \(n = 5\) only \(f_2(s), f_3(s)\) and \(f_4(s)\) give the result.

### 1.2 Determination of Perturbation Bounds

Given the polynomial of the type of Eqn. (1), determine the largest perturbation \(\varepsilon\) in the coefficients so that the family of polynomials

$$f(s) = s^n + \delta_n s^{n-1} + \cdots + \delta_0$$

where

$$a_i - \varepsilon < \delta_i < a_i + \varepsilon$$

is stable. This problem was solved by Barmish (1984), also with different weightings for the different coefficients. The strong Khariitonov theorem can be used and the Hurwitz matrices which correspond to the four Khariitonov polynomials are determined. The positivity of the leading principal minors of each Hurwitz matrix gives upper bounds for \(\varepsilon\). Then, \(\varepsilon_{\max}\) is equal to the minimum of the four upper bounds.

Another type of bound is given in Soh et al. (1985). Given the polynomial of Eqn. (1) which is represented by a point in the coefficient space, find the largest radius of a hypersphere so that the family of polynomials given by this hypersphere are Hurwitz. It is well known that the stability region in the coefficient space is bounded by two surfaces, one is an \((n-1)\)-dimensional hyperplane and the other is a union of \((n-2)\)-dimensional hyperplanes. Determining the distance between the given point and the two hyper-surfaces and taking the minimum gives the allowable radial perturbation.

### 2. Khariitonov Theorem for Complex Coefficients

In Khariitonov (1978) it is proved that the Hurwitz property of a set of interval polynomials having complex coefficients may be established from the Hurwitz property of eight extreme polynomials. Let

$$f(s) = \sum_{k=0}^{n} (a_k + j b_k) s^{n-k} \quad (a_k + j b_k \neq 0) \quad (4)$$

where

$$a_k \in [\tilde{a}_k, \bar{a}_k], \quad b_k \in [\tilde{b}_k, \bar{b}_k] \quad (5)$$

A necessary and sufficient condition that the set of interval polynomials defined in Eqs. (4) and (5) is Hurwitz is that the following eight extreme polynomials are Hurwitz.

508
Robust Stability of Interval Polynomials and Khariitonov’s Theorem

\[
\begin{align*}
 f_1(s) &= (a_0 + jb_0)s^n + (a_{n-1} + jb_{n-1})s^{n-1} + \\
 &+ (a_{n-2} + jb_{n-2})s^{n-2} + \\
 &+ (a_{n-3} + jb_{n-3})s^{n-3} + \cdots \\
 f_2(s) &= (\bar{a}_0 + j\bar{b}_0)s^n + (\bar{a}_{n-1} + j\bar{b}_{n-1})s^{n-1} + \\
 &+ (\bar{a}_{n-2} + j\bar{b}_{n-2})s^{n-2} + \\
 &+ (\bar{a}_{n-3} + j\bar{b}_{n-3})s^{n-3} + \cdots
\end{align*}
\]

Let \( f_1(s) = h_1(s) + s g_1(s) \) and \( f_2(s) = h_2(s) + s g_2(s) \) where \( h_1(s) \) and \( g_1(s) \) are the even and odd parts of \( f_1(s) \) respectively. A sufficient condition is that (a) \( f_1(s) \) and \( f_2(s) \) are Hurwitz, and (b) \( h_1(s) = h_2(s) \) or \( g_1(s) = g_2(s) \) (Bialas and Garloff 1985). Also

\[
f_{1,\delta}(s) = (1 - \lambda)h_1(s) + \lambda h_2(s) + s[(1 - \delta)g_1(s) + \delta g_2(s)]
\]

is Hurwitz for all \( \lambda, \delta \in [0, 1] \) iff \( f_1(s) \), \( f_2(s) \), \( f_3(s) = h_1(s) + s g_1(s) \) and \( f_4(s) = h_2(s) + s g_2(s) \) are Hurwitz.

A necessary and sufficient condition for Eqn. (9) to be Hurwitz is that \( f_3(s) \) is Hurwitz and the matrix \( \mathbf{H}^{-1} \mathbf{H} \) has no real negative eigenvalues where \( \mathbf{H} \) and \( \mathbf{H} \) are the Hurwitz matrices corresponding to the polynomials \( f_1(s) \) and \( f_3(s) \) (Bialas 1985).

3.2 Stability of a Polytope of Polynomials

An important result appeared in Bartlett et al. (1988) and came to be known as the edge theorem. A special version of the edge theorem can be stated as follows: a polytope is stable if and only if its exposed edges are stable. Here a polytope is the convex hull of a finite set of points in the coefficient space. This arises, for example, when the polynomial coefficients depend linearly on the uncertain parameters and when these parameters are confined to a polytopic set.

If \( \Omega \) is the set of points of the polytope in the coefficient space then the exposed edges are the one-dimensional intersections of \( \Omega \) and a hyperplane. This edge theorem means that it is sufficient to check only the exposed edges for the stability of the entire polytope. One method for checking the edges is by root locus. The method of Sect. 3.1 can also be used. Here only the stability of one corner of the polytope is needed at the beginning of the test. It is important to note that the edge theorem applies also to Schur stability.

4. Extensions to Discrete Systems

It was shown by counter examples that even the weak Khariitonov theorem does not hold for discrete systems. In the following a summary is given of different attempts to get Khariitonov-like results.

4.1 Result Based on the Bilinear Transformation

Given

\[
f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n \quad (a_0 \neq 0) \quad (10)
\]

with

\[
a_i \in [a_i, \bar{a}_i], \quad i = 0, 1, 2, \ldots, n \quad (11)
\]

the stability of Eqn. (10) subject to Eqn. (11) can be determined using the bilinear transformation (Bose and Zeheb 1986) \( z = (s+1)/(s-1) \) and applying the Khariitonov theorem. However, in this case the rect-

The original proof of Kharitonov is again based on the Hermite–Bieker theorem.

3. Convex Combinations of Polynomials and Polytopes of Polynomials

3.1 Convex Combinations of Stable Polynomials

Given two real polynomials \( f_1(s) \) and \( f_2(s) \)

\[
f_1(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n \quad (a_0 \neq 0) \quad (7)
\]

\[
f_2(s) = c_0 s^n + c_1 s^{n-1} + \cdots + c_{n-1} s + c_n \quad (c_0 \neq 0) \quad (8)
\]

under what condition is

\[
f_3(s) = (1 - \lambda)f_1(s) + \lambda f_2(s) \quad (9)
\]

Hurwitz for all \( \lambda \in [0, 1] \)?
Robust Stability of Interval Polynomials and Kharitonov's Theorem

The angular box given by Eqn. (11) shall be transformed to a box which is no more rectangular and has to be included in a larger rectangular box to apply the Kharitonov theorem. As a result we get only sufficient conditions for stability by this method.

4.2 Special Cases

In Hollett and Bartlett (1986) the special case is considered where \( g = \tilde{a} \), for \( i = 0, 1, 2, \ldots, \lfloor n \rfloor \) where \( \lfloor x \rfloor \) denotes the next lower integer if \( x \) is not an integer. In this case it is easy to prove that a necessary and sufficient condition for stability is that all the corners are stable.

In Mori and Kokame (1986) the following special case is considered

\[
f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \tag{12}
\]

with

\[|a_i| \leq b_i\]

that is, the limits of the coefficients are symmetric with respect to the origin \( (a_i = -\tilde{a}_i) \). In this case a necessary and sufficient condition for stability is \( \Sigma_{n,b_i<1} \).

4.3 Results for Low Order Systems

In Cieslik (1987) and Yeung and Wang (1987b) Kharitonov’s results were extended to discrete systems up to third order. It was shown that the stability of all the extreme polynomials (corner polynomials) is necessary and sufficient for the stability of the interval polynomials. This result was also demonstrated in Kraus et al. (1988). Moreover, they showed that for \( n=4 \) and \( n=5 \) stability can be concluded by considering the corner polynomials as well as possible supplementary points on some of the edges. These supplementary points, if existing, are obtained using the critical stability constraint.

4.4 Convex Combinations of Polynomials

The analog result of Bialas et al. (1985) can be stated as follows. Given Schur-stable \( f_1(z) \) and \( f_2(z) \)

\[
f_1(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \tag{13}
\]

\[
f_2(z) = c_0 z^n + c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n \tag{14}
\]

then \( f_\lambda(z) = (1-\lambda)f_1(z) + \lambda f_2(z) \) is Schur-stable for all \( \lambda \in [0, 1] \) if and only if the matrix \( S \Sigma S^{-1} \) has no real negative eigenvalues where \( S \) is the critical stability constraint, that is, the inner matrix for \( f_\lambda(z) \).

The analog result to the weak Kharitonov theorem

It is noted that the edge theorem is valid for discrete systems as well, so that investigating the stability of polytopes of polynomials can be reduced to investigating the stability of exposed edges by the above approach. Bartlett and Hollett (1988) investigated Schur-stability of polytopes of polynomials using bilinear transformation of the corners and the stability of the edges as in Bialas (1985).

4.5 The Counterpart of Kharitonov Theorem

The results in Kraus et al. (1988) can be extended to higher order systems using the edge theorem. Several possibilities exist: determining the possible supplementary points as in Kraus et al. (1988) or using the inner matrix for some of the edges.

4.6 Analog Results to Kharitonov Theorems

In Kraus, Anderson and Mansour (1988) an analog result to the weak Kharitonov theorem is derived. Let Eqn. (10) apply and let every \( a_i \) and \( a_{n-i} \) vary inside a region as shown in Fig. 1. If \( n \) is even \( a_{n/2} \) varies in an interval \([a_{n/2}, a_{n/2}^-]\). Then Eqn. (10) is stable for all values of \( a_i \) inside the prescribed region if and only if every member of the finite set of \( f(z) \) defined by every possible combination of corner points (and interval end-points in case \( n=2 \)) is stable. This result leads to several necessary and differing sufficiency conditions for the stability of polynomials where each \( a_i \) can vary independently in the interval \([a_i, \tilde{a}_i] \). As the sufficiency conditions become closer to necessity conditions the number of distinct polynomials for which stability has to be tested increases. The proof of these results depends on the discrete analog of the Hermite–Bielet theorem which can be stated as follows.

Theorem 3. \( f(z) \) has all roots in \(|z|<1\) if and only if the zeros of \( h(z) \) and \( g(z) \) are simple, all lie on \(|z|=1\) alternate on \(|z|=1\) and \(|a_i/a_j|<1\) where

\[
h(z) = \frac{1}{2} [f(z) + z^n f(1/z)]
\]

\[
g(z) = \frac{1}{2} [f(z) + z^n f(1/z)]
\]
Robust Stability of Interval Polynomials and Khartitonov’s Theorem

Thus

\[ h(z) = \frac{a_0 + a_n}{2} z^n + \frac{a_1 + a_{n-1}}{2} z^{n-1} + \cdots + \frac{a_1 + a_{n-1}}{2} z \]

\[ + \frac{a_0 + a_n}{2} \]

\[ = \alpha_0 z^n + \alpha_1 z^{n-1} + \cdots + \alpha_1 z + \alpha_0 \]  

(15)

\[ g(z) = \frac{a_0 - a_n}{2} z^n + \frac{a_1 - a_{n-1}}{2} z^{n-1} + \cdots + \frac{a_1 - a_{n-1}}{2} z \]

\[ - \frac{a_0 - a_n}{2} \]

\[ = \beta_0 z^n + \beta_1 z^{n-1} + \cdots - \beta_1 z - \beta_0 \]  

(16)

Note that \( \alpha_i \in [\alpha_i, \bar{\alpha}_i] \) and \( \beta_i \in [\beta_i, \bar{\beta}_i] \). Through projection of \( h(z) \) and \( g(z) \) on the real axis one gets \( h'(z) \) and \( g'(z) \) where the coefficients of \( \alpha_0, \alpha_1, \ldots, \beta_0, \beta_1 \ldots \) are Tchebychev and related polynomials. The line \([-1, 1]\) can be divided into intervals given by the different roots of these polynomials. The number of corners to be checked for stability is \( N = 40 \) where \( \delta \) is the number of intervals. This number of intervals increases less than quadratically while the total number of corners increases exponentially. For low order systems \((n < 6)\) a reduction similar to continuous systems is obtained.

Figure 2 shows the relation between system order \( n \) and the number of intervals \( N \). For \( n = 30 \), \( N = 4 \times 144 = 576 \) (out of \( 2^{31} \) corners). In Mansour et al. (1988) these results are demonstrated and considered as the discrete analog of the strong Khartitonov theorem.

Figure 2

The relation between system order \( n \) and the number of intervals \( \delta \)

\[ f_1(j\omega) = h(j\omega) + j\omega g(j\omega) \]

\[ f_2(j\omega) = \bar{h}(j\omega) + j\omega \bar{g}(j\omega) \]

\[ f_1(j\omega) = \bar{h}(j\omega) + j\omega \bar{g}(j\omega) \]

\[ f_2(j\omega) = h(j\omega) + j\omega g(j\omega) \]

which are exactly the four Khartitonov polynomials. These are illustrated in Fig. 3.

It is easy to prove using Cremer–Leonhard–Michailow criterion that a necessary and sufficient condition for robust Hurwitz stability is that the corner polynomials \( f_1, f_2, f_3, f_4 \) are Hurwitz. It is to be noted that for different frequencies the rectangle is shifted but the origin never appears on its boundary. Using the same Cremer–Leonhard–Michailow criterion it is trivial to show that for \( n = 3, 4, 5 \) one needs to check only \( f_1, f_2, f_3, f_4 \) respectively. In Argoun (1987), conditions on the frequency response of the even and odd parts of a perturbed polynomial to remain Hurwitz are given. These conditions allow freedom in allocating different weights to various coefficients to reflect different levels of uncertainty.

5. Khartitonov Results in the Frequency Domain

5.1 Robust Hurwitz Stability

According to the strong Khartitonov theorem one needs to check only four corners for the stability of

\[ f(s) = \sum_{k=0}^{n} a_k s^{a_k} \quad (0 < a_0 \leq a_n \leq a_{n-1}) \]

\[ f(s) = h(s) + s g(s) \]  

(17)

where

\[ h(s) = a_n + a_{n-1} s^2 + a_{n-2} s^4 + \cdots \]  

(18)

\[ g(s) = a_{n-1} + a_{n-2} s^2 + a_{n-3} s^4 + \cdots \]  

(19)

This result can be easily obtained in the frequency domain. Letting \( s = j\omega \) we get \( h(j\omega) = \text{Re}[f(j\omega)] \) and \( g(j\omega) = \text{Im}[f(j\omega)] \).

It was noticed that the box of the coefficient space is mapped for a fixed frequency into a rectangle in the complex plane \( f(j\omega) \). The corners of this rectangle are

Figure 3

Khartitonov results in the frequency domain

511
5.2 Robust Schur Stability

The analog of the strong Kharitonov theorem for discrete systems described in Sect. 4.6 can be easily obtained using frequency response methods through mapping on the unit circle. Let

\[ f(z) = \sum_{k=0}^{n} a_k z^{-k} \]
\[ h(z) = \frac{1}{2} \left[ f(z) + z^n f(1/z) \right] \]
\[ g(z) = \frac{1}{2} \left[ f(z) - z^n f(1/z) \right] \]

thus

\[ h(z) = \alpha_0 z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n + \alpha_0 \]
\[ g(z) = \beta_0 z^n + \beta_1 z^{n-1} + \ldots - \beta_n - \beta_0 \]

For even \( n = 2\gamma \)

\[ h(e^{i\theta}) = 2e^{i\theta n/2} \left[ a_0 \cos \gamma \theta + a_1 \cos(\gamma - 1)\theta + \ldots + a_{\gamma-1} \cos(0.5\theta) \right] \]
\[ g(e^{i\theta}) = 2e^{i\theta n/2} \left[ \beta_0 \sin \gamma \theta + \beta_1 \sin(\gamma - 1)\theta + \ldots + \beta_{\gamma-1} \sin(0.5\theta) \right] \]

and for \( n = 2\gamma - 1 \)

\[ h(e^{i\theta}) = 2e^{i\theta n/2} \left[ a_0 \cos(\gamma - 0.5)\theta + a_1 \cos(\gamma - 1.5)\theta + \ldots + a_{\gamma-1} \cos(0.5\theta) \right] \]
\[ g(e^{i\theta}) = 2e^{i\theta n/2} \left[ \beta_0 \sin(\gamma - 0.5)\theta + \beta_1 \sin(\gamma - 1.5)\theta + \ldots + \beta_{\gamma-1} \sin(0.5\theta) \right] \]

The mapping of the rectangular box of the \( \alpha-\beta \) space in the complex plane is shown in Fig. 4.

![Figure 4](image-url)

The corner points of the rectangle correspond to

\[ \bar{h}^*, \bar{g}^* \leftrightarrow f_1 \]
\[ h^*, \bar{g}^* \leftrightarrow f_2 \]
\[ h^*, g^* \leftrightarrow f_3 \]
\[ \bar{h}^*, g^* \leftrightarrow f_4 \]

For an interval where the sign of the trigonometric functions does not change, the corners of the rectangle correspond to four fixed corners of the box in the space. It is easy to see that the number of corners to be checked \( N = 4\delta \) where \( \delta \) are the number of intervals of \( \theta \) which correspond to the result in Sect. 4.6.

The number of intervals \( \delta \) is given by

\[ \delta = 1 + \sum_{k=3, 5, 7, \ldots, n} \phi(k) \quad (\text{for } n \text{ odd} \geq 3) \]

and

\[ \delta = 1 + \sum_{k=4, 6, 12, \ldots, n} \phi(k) \quad \text{or } (n-2) \]
\[ + 2 \sum_{k=6, 10, 14, \ldots, n-2} \phi(k) \quad (\text{for } n \text{ even} \geq 2) \]

where \( \phi(k) \) is the Euler function.

5.3 Extensions in the Frequency Domain

Considering other regions in the \( s \)-plane \( \Gamma \) with boundary of other than the imaginary axis or the unit circle and considering a polytope in the coefficient space it can be shown that the mapping of the polytope for any \( s \) is a polygon in the complex plane. The following theorem gives an important result.

**Theorem 4.** The necessary and sufficient condition for the \( \Gamma \)-stability of the polytope of polynomials is that one of the corner polynomials is \( \Gamma \)-stable and the polygon in the complex plane for all \( s \in \partial \Gamma \) does not include the origin.

Through parametrization and computation of a single function robust stability of the same problem can be obtained (Barmish 1988). In Dasgupta et al. (1988) a generalization of the edge theorem which extends the results to polytopes of functions that are not necessarily polynomials is considered.

*See also: Linear Systems: Robustness (Suppl. 1)*
Bibliography


Argoun M B 1987 Stability of a Hurwitz polynomial under coefficient perturbations: Necessary and sufficient conditions. *Int. J. Control* 48(2), 739–44


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