HYBRID COMPUTATIONAL METHOD FOR LIMIT CYCLES IN NONLINEAR SAMPLED-DATA AND PULSE-WIDTH MODULATED CONTROL SYSTEMS

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1. Introduction

Different modi of limit cycles can exist in nonlinear sampled-data control systems such as the system shown in Fig. 1 [1],[2]. It is possible using an analytical method [2],[3] to detect the different oscillations which can exist in such systems. A digital computer program was developed in [3],[4] which allows the calculation of the different oscillations in a nonlinear sampled-data control systems with general piecewise linear element and general second order system. There is no problem in applying the method to higher order systems. The information we get from the analytical method is complete in the sense that if the oscillation exists it is certainly determined.

Another method which is called "simple search method" was first introduced in [2]. This method is based on the fact that it is possible to reach the required self-sustained oscillation from an unknown region in the state and if we start from initial states outside the region we reach another limit cycle or go to an equilibrium state. The question is whether it is possible to start from anywhere in the state space and reach the particular limit cycle which we look for. It is possible to solve this question by using the simple search method in [2] or the search method based on Newton-Raphson iteration formula in [3]. This is on the condition that the iteration converges. This depends on how complex is the topology in the state space of the system in question.

2. Simple iteration method

If we start from anywhere in the state space at \( x(0) \) and integrate for a time \( T \) which is the period of oscillation, we reach a final state \( x(T) \). The new initial state vector is calculated using a simple iteration formula which satisfies the condition that if the system reaches the required limit cycle it will remain there.

Let \( x_p \) be the initial state vector after \( p \) iteration steps and \( x_p^* \) the corresponding final state vector after period \( T \). For a limit cycle of period \( T \) to exist the relation

\[
\hat{F}(x_p) = x_p - x_p^* \tag{1}
\]

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should tend to zero as $p$ tends to infinity.
For a symmetrical limit cycle it is sufficient to integrate for half the period of oscillation $\frac{T}{2}$.

In this case

$$F(x_p) = x_p + z_p$$ (2)

should tend to zero as $p$ tends to infinity.
With this formulation the problem is reduced to a problem in nonlinear algebra. A simple iteration formula is

$$x_{p+1} = x_p - \frac{1}{2} F(x_p)$$ (3)

Fig. 2 and Fig. 3 show the results of a hybrid computer program for a relay-sampled-data control system.

3. Newton-Raphson iteration formula

The convergence of the search method can be accelerated if we use Newton-Raphson iteration method.

Expanding $F(x)$ in a Taylor series we get

$$F(x_{p+1}) = F(x_p) + \frac{\partial F}{\partial x} (x_{p+1} - x_p)$$ (4)

+ higher order terms

$\frac{\partial F}{\partial x}$ is the Jacobian matrix $J$ given by

$$J = \begin{bmatrix}
F'_1 x_1 & F'_1 x_2 & \ldots & F'_1 x_n \\
F'_2 x_1 & F'_2 x_2 & \ldots & F'_2 x_n \\
\vdots & \vdots & \ddots & \vdots \\
F'_n x_1 & F'_n x_2 & \ldots & F'_n x_n
\end{bmatrix}$$ (5)

The subscripts $x_1, x_2, \ldots, x_n$ denote the partial derivatives at $x_p$

Setting $F(x_{p+1}) = 0$ gives

$$F(x_p) + J(x_{p+1} - x_p) = 0$$ (6)
Solving this equation for $X_{p+1} - X_p$ we get

$$X_{p+1} = X_p - J^{-1}(Z_p)$$  \hspace{1cm} (7)

Comparing the formula (7) with the simple iteration formula (1) we see that the factor $2$ is replaced by the Jacobian $J$.

For the calculation of the partial derivatives we use the system state transition equation

$$\overset{N}{Z}(k+1) = B_{x}(k) + B_{u}(k)$$  \hspace{1cm} (8)

where $B$ and $Q$ are functions of the sampling period $T_s$.

For the method: $\left(\overset{N}{Z} = MT_{g}\right)$

$$\overset{N}{Z}(k+M) = B_{x}(k) + B_{u}(k) + \ldots + B_{u}(k+M-2) + B_{u}(k+M-1)$$  \hspace{1cm} (9)

The control variable $u$ is a nonlinear function of the state vector. However, we can neglect all the terms in equation (9) containing the control variable. In this case we get

$$Z(k+1) = Z(k) - (I - B_i)Z_p + \text{terms containing } u$$  \hspace{1cm} (10)

which yields

$$J = I - B_i$$  \hspace{1cm} (11)

Substituting in equation (7) we get the iteration formula for the method

$$X_{p+1} = X_p - (I - B_i)^{-1}Z_p$$  \hspace{1cm} (12)

For the $\frac{N}{2}$ - method we get

$$\overset{N}{Z}(k+\frac{N}{2}) = B_{x}(k) + \ldots + B_{x}(k+\frac{N}{2})$$

$$Z(k) = Z_p + \overset{N}{Z}_p = (I + B_i)Z_p + \text{terms containing } u$$  \hspace{1cm} (14)

which yields

$$J = I + B_i$$  \hspace{1cm} (15)

and the iteration formula for the $\frac{N}{2}$ - method will be

$$X_{p+1} = X_p - (I + B_i)^{-1}Z_p$$  \hspace{1cm} (16)

For a second order system the iteration formulas will be

$$x_{1p+1} = x_{1p} + \frac{F_2 F_1 x_2 - F_1 F_2 x_1}{F_1 x_1 F_2 x_2 - F_1 x_2 F_2 x_1}$$

$$x_{2p+1} = x_{2p} + \frac{F_1 F_2 x_1 - F_2 F_1 x_1}{F_1 x_1 F_2 x_2 - F_1 x_2 F_2 x_1}$$  \hspace{1cm} (17)

Fig. 4 and Fig. 5 show the results of a hybrid computer program for a relay-sampled-data control system.

Fig. 6 and Fig. 7 show results when the relay has a deadzone \cite{5}.
4. Hybrid computation

A general block diagram of the hybrid computation of nonlinear sampled-data control systems is shown in Fig. 8. The process together with the nonlinearity are simulated on the analog computer where as the digital computer performs the following operations:

a) potentiometer setting
b) control of mode of operation of the analog computer
c) calculation of the state transition matrix and the partial derivatives in case of using Newton-Raphson iteration formula
d) calculation of the new initial conditions using the iteration formula.
A simplified flow chart for the simple search method is shown in Fig. 9 while Fig. 10 shows the flow chart for the search method based on Newton-Raphson iteration formula.

5. Limit cycles in pulse-width modulated control systems by the search method.

It is possible to apply the two search methods represented above to pulse-width modulated control systems. The block diagram representing the system is shown in Fig. 11. The pulse-width modulator is preceded by a saturation non-linearity. This block diagram can represent e.g. a direct current motor controlled by thyristors in the armature circuit (pulsed d.c. motor). Fig. 12 shows a general block diagram of the hybrid computation of the PDM-system. The process together with the nonlinearity are simulated on the analog computer while the digital computer performs the pulse-width modulation together with the operations mentioned in connection with nonlinear sampled-data control systems.
A simplified flow chart for the simple search method as well as for the search method based on Newton-Raphson iteration formula are shown in Fig. 13 and Fig. 14 respectively.
Fig. 15 and Fig. 16 show some results on the hybrid computer for a second order system [6].

6. Analytical determination of limit cycles in pulse-width modulated control systems.

Equation (8) is the state transition equation of a sampled-data control system. For a pulse-width modulated control system each period is divided into two parts. During the first part the control variable is a pulse of duration \( \tau \) proportional to the absolute value of the error \( e \). During the second part the control variable is zero. The pulse-width \( \tau \) cannot exceed the period \( T_s \) so that a saturating element can be used prior to the pulse-width modulation.

Accordingly we can write the following transition equations:

\[
\begin{align*}
  \dot{z}(k + 1) &= B(t_\tau)z(k) + B(t_\tau)u(k) \\
  \dot{z}(k + 1) &= B(t_\tau)z(k) + B(t_\tau)u(k) \\
  \dot{z}(k + 1) &= \beta(\tau - T_s)\beta(T_s)z(k) + B(\tau - T_s)u(k) \\
  \dot{z}(k + 1) &= \beta(\tau - T_s)\beta(T_s)z(k) + B(\tau - T_s)u(k)
\end{align*}
\]

For asymmetrical limit cycles

\[
\dot{z}(k+1) = \begin{cases} \dot{z}(k) + B(MT_s)z(k) & \text{for positive \( \tau \)} \\
  \dot{z}(k) + B(MT_s - T_s)z(k) + B(\tau - T_s)u(k) & \text{for negative \( \tau \)} \end{cases}
\]
+B(j\tau_{k-M})B(j\tau_{k-M-1})u(k+M-1)

E(BM_\phi)z(k) + \sum_{j=1}^{n} E(Bj\tau_{k-M})B(j\tau_{k-M-1})u(k+M-1)

[I-B(M_\phi)]z(k) = \sum_{j=1}^{n} E(Bj\tau_{k-M})B(j\tau_{k-M-1})u(k+M-1)

z(k)[I-B(M_\phi)]^{-1} \sum_{j=1}^{n} E(Bj\tau_{k-M})B(j\tau_{k-M-1})u(k+M-1)

where \tau_{k+1} = K.abs.[x(k+1)]

when K.abs.[x(k+1)] \leq \tau_\phi

\tau_{k+1} = \tau_\phi

when K.abs.[x(k+1)] > \tau_\phi

and u(k+1) = -sign.[x(k+1)]

Similarly, for symmetrical limit cycles we get

\frac{M_\phi}{2}

x(k)[I+B(\frac{M_\phi}{2})]^{-1} \sum_{j=1}^{n} E(Bj\tau_{k-M})B(j\tau_{k-M-1})u(k+M-1)

with \tau_{k+1} and u(k+1) as above.

7. Conclusion

In this paper a hybrid computational method which permits the computation of the different modes of oscillations in a nonlinear sampled-data control systems is represented.

The same method is used to compute the oscillations in nonlinear pulse-width modulated control systems. The limit cycles in the later systems can be also calculated using the analytical method.

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Limit cycles in nonlinear sampled-data control systems.


Hybridrechenmethode für Grenzyklen in nichtlinearen und pulsbreitenmodulierten Systemen.


Grenzyklen in nichtlinearen, getasteten pulsbreitenmodulierten und phasenmodulierten Systemen.