SYNCHRONIZATION OF "n" INTEGRAL-PLUS-TIME CONSTANT PLANTS WITH NON-IDENTICAL GAIN AND TIME CONSTANTS

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Abstract. Synchronization of "n" integral-plus-time constant plants with non-identical gain and time constants, non-identical outputs initial conditions (i.e., \( G_i(s) = \frac{K_i}{s(s+a_i)} \), \( y_i(0) = \alpha_i \), \( \dot{y}_i(0) = \beta_i \), \( i = 1, 2, \ldots, n \)) will be studied in this paper. Achievement of two objectives will fulfill our synchronization goal: (i) Making all the "n" steady-state outputs identical. (ii) Having each one of the identical steady-state outputs time integral of a common input \( x(t) \). Two examples, being simulated on analog computer, will, most vividly, support the theoretical results.

Key Words—Feedback control systems, initial-value problem, modelling, multiple-input-multiple-output systems, stability, synchronization.

1. Introduction

As has been previously discussed (Unbehauen and Vakilzadeh, 1989 b), for high-precision scientific research works, as well as on production line of many goods-producing plants, where a number of "assumedly" identical machines with identical inputs are commissioned, then, naturally, one would expect identical outputs from them. However, if the outputs initial conditions of these machines are not identical (external disturbances) and/or their input-output characteristics differ from one another — due to original mismatches in their corresponding components or uneven aging effects on them — (internal disturbances), then these machines will offer "different" outputs when they are excited from the "same" source.

The most celebrated of these machines, both from theoretical and practical points of view, is the integral-plus-time constant plant: \( [K/s(s+a)] \), which is, say, the transfer function of an armature-controlled DC-motor with negligible armature inductance — with or without viscous friction damping (Ogata, 1970). If these machines, or plants, are completely identical except for their outputs initial conditions, i.e.,

\[
G_i(s) = \frac{K_i}{s(s+a)} \quad \text{or} \quad \begin{cases}  
G_i(s) = \frac{K_i}{s(s+a)} \\
 y_i(0) = \alpha_i \\
 \dot{y}_i(0) = \beta_i 
\end{cases}, \quad i = 1, 2, \ldots, n,
\]

* Received by the editors February 16, 1989 and in revised form July 21, 1989.
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it had been shown (Unbehauen and Vakilzadeh, 1988 a) that if, once and for all, the plants are interconnected through identical P-controllers, which were taken as unity, then

(i) The MIMO system would be stable $\forall n, K, a$,

$$y_i(t) = y_2(t) = \cdots = y_n(t) \mid_{t \to \infty} \forall x(t),$$

(ii) $$y(t) \mid_{t \to \infty} = \left[ \mathcal{L}^{-1} \left( \frac{K}{s(s+a)}X(s) \right) \right]_{t \to \infty} + c, \quad i = 1, 2, \ldots, n,$$

where

$$c = \frac{1}{n} \left[ \sum_{m=1}^{n} \left( a_m + \frac{1}{a} \beta_m \right) \right].$$

Result (ii) shows that in the steady state the errors between the "n" outputs, which were entirely due to different outputs initial conditions, had been eliminated; while result (iii) indicates that in the steady state the operating action of each plant, which is integration with the integrating coefficient of: $[K/a]$, had been preserved – as was originally desired.

The case of "n" integral-plus-times constant plants with non-identical gain constants and different outputs initial conditions, i.e.,

$$G_i(s) = \frac{K_i}{s(s+a)}$$

$$y_i(0+) = a_i, \quad i = 1, 2, \ldots, n, \quad K_1 \leq K_2 \leq \cdots \leq K_n$$

has also been thoroughly studied (Unbehauen and Vakilzadeh, 1989 b), with the main objectives that:

The "n" steady-state outputs be identical.

These identical steady-state outputs be "time integral" of the common inputs $x(t)$.

Contrary to identical plants just mentioned, in this case we observed that for "each" type of common inputs $x(t)$ the plants must be interconnected through a "different" type of identical controllers $H(s)$ – which will, inevitably, poses the formidable task of ascertaining the stability of the MIMO system. Invoking upon the results of non-identical simple-integral plants (Unbehauen and Vakilzadeh, 1989 a), it was shown how elegantly, and simply, the stability problem can be circumvented through consideration of only the value of smallest gain constant ($K_1$) and the number of plants ($n$).

2. Problem Formulation

As can easily be observed (Unbehauen and Vakilzadeh, 1989 b), the study of synchronization of integral-plus-time constant plants with non-identical gain constants ($K_i, i = 1, 2, \ldots, n$) rests heavily on the technique of interconnecting the "n" plants through identical controllers $H(s)$ and also the results obtained
"n" integral-plus-time constant plants

for non-identical simple-integral plants (Unbehauen and Vakilzadeh, 1989 a). Unfortunately, this technique fails its usefulness in case of non-identical times constants \((a_i, \ i = 1, 2, \ldots, n)\) - whether the gain constants are equal or not!

In this paper, we shall adopt a new method and will consider the most general case, i.e.,

\[
G_i(s) = \frac{K_i}{s(s+a_i)}
\]

\[
\begin{align*}
\dot{y}_i^{(n)}(0^+) &= \alpha_i \\
\ddot{y}_i^{(n)}(0^+) &= \beta_i
\end{align*}
\]

with

\[
x(t) = \frac{1}{q!} t^q, \quad q = 0, 1, 2, \ldots,
\]

such that the following objectives are fulfilled:

\[(i) \quad \dot{y}_1^{(n)}(t) = \dot{y}_2^{(n)}(t) = \cdots = \dot{y}_{n-1}^{(n)}(t) = \dot{y}_n^{(n)}(t) |_{t \to \infty} = \dot{y}_n^{(n)}(t) |_{t \to \infty}, \quad (A)\]

\[(ii) \quad y_n^{(n)}(t) |_{t \to \infty} = \frac{K_n}{a_n} \int x(t)dt + \int \dot{y}_n^{(n)}(t, K_n, a_n) + f^{(n)}(\alpha_n, \beta_n, a_n), \quad (B)\]

where, respectively, \(K_n, a_n, \alpha_n, \beta_n\) are the gain constant, time constant, first and second initial conditions of the plant which has been singled out to act as "Master Plant", which from now on will be shown on our diagram as (M); while the remaining "\(n-1\)" plants are named "Slave Plants", which will be characterized by the letter (S).

Note:
---
As for the last two references in the list of references, the symbol (·) over an alphabetic letter implies the number of plants in our system. For example, \(Y_1^{(3)}(s)\) means the transform of "1st" output of a "3-plant" system.
---
The suffix "i" for the time constants (as) and gain constants (Ks) does not designate hierarchy for either of them. For example, it is not necessary that \(a_3 > a_2\) or \(K_4 < K_5\).
---
If the time constant of one plant is greater than that of another plant, then it is not necessary that the corresponding gain constants follow suit - and vice versa. For instance,

\[a_3 > a_5 \iff K_3 > K_6.\]

In simple language: numbers 1, 2, ..., \(n\) are used only for labelling these machines (or plants) and have no other significance.

So, objective (i), or Eq. (A), requires that during the transient time the errors between the outputs of the "nth" plant, i.e., "Master Plant" (M), and the remaining "\(n-1\)" plants, i.e., "Slave Plants" (S), to be eliminated (the closed-loop phenomenon); while objective (ii), or Eq. (B), commands that in the steady state the "\(n\)" plants to behave as identical integrators, with integrating coefficient of: \([K_n/a_n]\) (the open-loop characteristic).
3. Analysis for "n-Plant" System

Figure 1 shows the block diagram for synchronization of an "n-plant" system. Note that:
— The "Slave Plants" (S) have no effect on the "Master Plant" (M).
— There is no interaction whatsoever between the "Slave Plants" (S).

The transform matrix equation of Fig. 1 is given by

\[
\begin{bmatrix}
  h_1(s) & 0 & \cdots & -K_1H(s) \\
  0 & h_2(s) & \cdots & -K_2H(s) \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & h_n(s)
\end{bmatrix}
\begin{bmatrix}
  Y_1^{(n)}(s) \\
  Y_2^{(n)}(s) \\
  \vdots \\
  Y_n^{(n)}(s)
\end{bmatrix}
= \begin{bmatrix}
  K_1 \\
  K_2 \\
  \vdots \\
  K_n
\end{bmatrix}X(s)
+ \begin{bmatrix}
  (s+a_1)\alpha_1 \\
  (s+a_2)\alpha_2 \\
  \vdots \\
  (s+a_n)\alpha_n
\end{bmatrix}
+ \begin{bmatrix}
  \beta_1 \\
  \beta_2 \\
  \vdots \\
  \beta_n
\end{bmatrix},
\]

where

Fig. 1. Block diagram for synchronization of "n" integral-plus-time constant plants with non-identical gain and time constants and non-identical outputs initial conditions.


\[ h_i(s) = s^2 + a_i s + K_i H(s), \quad i = 1, 2, \ldots, n - 1, \]
\[ h_i(s) = s^2 + a_n s. \]

From the above matrix equation, we get

\[ Y_i^{(n)}(s) = \frac{K_i N_i^{(n)}(s)}{s [s + a_n] D_i^{(n)}(s)} X(s) + \frac{N_i^{(n)}(s)}{s D_i^{(n)}(s)} + \frac{N_i^{(n)}(s)}{s [s + a_n] D_i^{(n)}(s)}, \]
\[ i = 1, 2, \ldots, n - 1, \tag{1} \]

\[ Y_n^{(n)}(s) = \frac{K_n}{s [s + a_n]} X(s) + \frac{\alpha_n}{s} + \frac{\beta_n}{s [s + a_n]}, \tag{2} \]

where

\[ D_i^{(n)}(s) = s^2 + a_i s + K_i H(s), \tag{3a} \]
\[ N_i^{(n)}(s) = s^2 + a_i s + K_i H(s), \tag{3b} \]
\[ N_i^{(n)}(s) = s [s + a_i] \alpha_i + K_i H(s) \alpha_n, \tag{3c} \]
\[ N_i^{(n)}(s) = s [s + a_i] \beta_i + K_i H(s) \beta_n, \tag{3d} \]

\[ i = 1, 2, \ldots, n - 1. \]

Let

\[ E_i^{(n)}(s) = Y_i^{(n)}(s) - Y_n^{(n)}(s) \]
\[ = \frac{[K_i - K_n] s + [K_i a_n - K_n a_i]}{s + a_n} D_i^{(n)}(s) X(s) \]
\[ + \frac{s + a_i}{D_i^{(n)}(s)} \left[ \alpha_i - \alpha_n \right] + \frac{s + a_n}{s + a_n} D_i^{(n)}(s) \]
\[ = \{ E_i^{(n)}(s) \} + \{ E_i^{(n)}(s) \}, \quad i = 1, 2, \ldots, n - 1. \tag{4} \]

So, \( E_i^{(n)}(s) \) is the error due to common inputs \( X(s) \) and \( E_i^{(n)}(s) \) being the error due to initial conditions. Let us now study various types of common inputs \( X(s) \) and the corresponding controller \( H(s) \) for fulfilment of Eqs. (A) and (B).

3.1 **Zero-input response \( x(t) = 0 \)**

Let

\[ X(s) = 0, \tag{5} \]
\[ H(s) = A_0 \text{ (P-controller).} \tag{6} \]

Substituting \( H(s) = A_0 \) in Eq. (3a) we see that the \( 2(n-1) \) root of \( D_i^{(n)}(s) \) lie in the left-half of s-plane, \( \forall A_0 \). Therefore, from Eq. (4), we have

\[ e_i^{(n)}(t) \bigg|_{t \to \infty} = e_i^{(n)}(t) \bigg|_{t \to -\infty} = 0 \Rightarrow y_i^{(n)}(t) \bigg|_{t \to -\infty} = y_n^{(n)}(t) \bigg|_{t \to -\infty}, \quad \forall A_0, \]
\[ i = 1, 2, \ldots, n - 1, \]
because \( e^{a}(t) = 0 \). Using Eq. (2), we get

\[
y_{i}^{(n)}(t) \bigg|_{t \rightarrow \infty} = y_{n}^{(n)}(t) \bigg|_{t \rightarrow \infty} = f^{(n)}(a_n, \beta_n, a_n),
\]

\( i = 1, 2, \ldots, n-1, \)

where

\[
f^{(n)}(a_n, \beta_n, a_n) = a_n + \frac{1}{a_n} \beta_n.
\]

(7)

### 3.2 Impulse response \([x(t) = \delta(t)]\)

Let

\[
\begin{align*}
X(s) &= 1 \text{ (unit impulse input)} \\
H(s) &= A_0
\end{align*}
\]

(8)

substituting Eqs. (6) and (8) in Eq. (4), we have

\[
e^{i_{in}^{(n)}(t)} \bigg|_{t \rightarrow \infty} = e^{a_{i}(t)} + e^{2_{in}^{(n)}(t)} \bigg|_{t \rightarrow \infty} = 0 + 0
\]

\[
\Rightarrow y_{i}^{(n)}(t) \bigg|_{t \rightarrow \infty} = y_{n}^{(n)}(t) \bigg|_{t \rightarrow \infty}, \quad \forall A_n,
\]

\( i = 1, 2, \ldots, n-1, \)

and from Eq. (2), we have

\[
y_{i}^{(n)}(t) \bigg|_{t \rightarrow \infty} = y_{n}^{(n)}(t) \bigg|_{t \rightarrow \infty}
\]

\[
= \frac{K_n}{a_n} + f^{(n)}(a_n, \beta_n, a_n)
\]

\[
= \frac{K_n}{a_n} \int_{t}^{\infty} \delta(t) \, dt + f^{(n)}(a_n, \beta_n, a_n),
\]

\( i = 1, 2, \ldots, n-1 \)

So,

- The MIMO system is stable \( \forall n, K_i, K_n, a_i, a_n, A_0 \) \( i = 1, 2, \ldots, n-1 \).
- \( y_{1}^{(n)}(t) = y_{2}^{(n)}(t) = \cdots = y_{n-1}^{(n)}(t) \bigg|_{t \rightarrow \infty} = y_{n}^{(n)}(t) \bigg|_{t \rightarrow \infty} \)

\[
= \frac{K_n}{a_n} \int_{t}^{\infty} \delta(t) \, dt + f^{(n)}(a_n, \beta_n, a_n)
\]

if \( x(t) = 0 \)

\[
= \frac{K_n}{a_n} \int_{t}^{\infty} \delta(t) \, dt + f^{(n)}(a_n, \beta_n, a_n)
\]

if \( x(t) = \delta(t) \)

\[\Rightarrow\] fulfilment of Eqs. (A) and (B)

where \( f^{(n)}(a_n, \beta_n, a_n) \) has the value given by Eq. (7).

### 3.3 Step response \([x(t) = u(t)]\)

Let

\[
\begin{align*}
X(s) &= \frac{1}{s} \\
H(s) &= A_0 + A_1 \frac{1}{s}
\end{align*}
\]

(9)

(10)
substituting Eqs. (9) and (10) in Eq. (4), we have

$$E_{n}^{(n)}(s) = \left( \frac{[K_1-K_n]s + [K_n a_n - K_n a_i]}{s + a_n D_i^{(n)}(s)} \right)$$

$$+ \left\{ \frac{s[s+a_i]}{D_i^{(n)}(s)} \left[ \alpha_i - \alpha_n \right] + \frac{s[s+a_i][\beta_i - s(s+a_i)\beta_n]}{s+a_n D_i^{(n)}(s)} \right\},$$

where

$$D_i^{(n)}(s) = s^3 + a_i s^2 + K_i A_i s + K_i A_1, \quad i = 1, 2, \ldots, n-1.$$  \hspace{1cm} (11)

Applying Routh-Hurwitz stability criterion to Eq. (11), we can say that all the $3(n-1)$ roots of $D_i^{(n)}(s)$ lie in the left-half of $s$-plane if

$$A_1 < (a_1) A_0$$
$$A_1 < (a_2) A_0$$
$$\vdots$$
$$A_1 < (a_{n-1}) A_0$$  \hspace{1cm} (12)

where

$$a_{min} = \min\{a_1, a_2, \ldots, a_{n-1}\}. \hspace{1cm} (13)$$

Figure 2 shows the admissible region for parameter $A_1$ in relation to parameter $A_0$. What values for $A_0$ and $A_1$ should be chosen can be determined through minimization, say, of total integral square-error, by using Eq. (4). We shall not involve ourselves here with this topic, Unbehauen and Vakilzadeh (1988 a; b) will give adequate insight to the optimization problem. Let us reiterate that “$a_{min}$” in Eq. (13) implies the smallest value of the time constants of the $(n-1)$ “Slave Plants” – whichever plant that belongs to!

Now, if inequality (12) is satisfied, then

![Admissible Region for $A_1$](image_url)

Fig. 2. Admissible region for parameter $A_1$ in relation to the parameter $A_0$. (For the PI-controllers: $H(s) = A_0 + A_1 (1/s)$ and PI2-controllers: $H(s) = A_0 + A_1 (1/s) + A_2 (1/s^2)$.)
\[
e^{(n)}_{in}(t) \bigg|_{t=\infty} = e^{(n)}(t) + e^{(n)}_{in}(t) \bigg|_{t=\infty} = 0 + 0
\]

\[
\Rightarrow y^{(n)}_i(t) \bigg|_{t=\infty} = y^{(n)}_m(t) \bigg|_{t=\infty},
\]

\[i = 1, 2, \cdots, n-1,
\]

and from Eq. (2), we get

\[
y^{(n)}_i(t) \bigg|_{t=\infty} = y^{(n)}_n(t) \bigg|_{t=\infty}
\]

\[
= \frac{d}{ds} \left( \frac{K_n}{s+a_n} \exp(st) \right)_{s=t} + f^{(n)}(a_n, \beta_n, a_n)
\]

\[
= \frac{K_n}{a_n} t + f^{(n)}_0(t, K_n, a_n) + f^{(n)}(a_n, \beta_n, a_n)
\]

\[
= \frac{K_n}{a_n} \int u(t)dt + f^{(n)}_0(t, K_n, a_n) + f^{(n)}(a_n, \beta_n, a_n),
\]

\[i = 1, 2, \cdots, n-1
\]

where

\[
f^{(n)}_0(t, K_n, a_n) = -\frac{K_n}{a_n}
\]

and, again, \(f^{(n)}(a_n, \beta_n, a_n)\) has the value given by Eq. (7). So,

- The MIMO system is stable \(\forall n, K_i, K_n, a_i, a_n, A_0, A_1 < (a_{min})A_0 \) \(i = 1, 2, \cdots, n-1\).
- \(y^{(n)}_1(t) = y^{(n)}_2(t) = \cdots = y^{(n)}_n(t) \bigg|_{t=\infty} = y^{(n)}_m(t) \bigg|_{t=\infty}
\]

\[= \frac{K_n}{a_n} \int u(t)dt + f^{(n)}_0(t, K_n, a_n) + f^{(n)}(a_n, \beta_n, a_n)
\]

\[\Rightarrow \text{fulfilment of Eqs. (A) and (B).}
\]

Some elaboration is merited at this point, since step input is the most practical type of input – while other types have only academic importance. Let us assume that the “\(n\)” integral-plus-time constant plants,

\[
\left[ \frac{K_1}{s(s+a_1)}, \frac{K_2}{s(s+a_2)}, \cdots, \frac{K_n}{s(s+a_n)} \right]
\]

be field-controlled dc-motors with different outputs initial conditions and with common inputs \(x(t)\). If, with \(x(t) = u(t)\), only synchronization of “speeds” of these motors is the prime objective, then all the \((n-1)\) identical \(H(s)\)-controllers (see Fig. 1) can be only of “P-type”, i.e., \(H(s) = A_0\). Having chosen so, then all the steady-state “speeds” would have “identical” values of: \([K_n/a_n]\), \(\forall A_0\) – where \(K_n\) and \(a_n\) are, respectively, the gain constant and time constant of the “Master Plant” (M). From the available information about these machines one, naturally, would choose that machine to act as “Master Plant” which has the “nearest” value of \([K_n/a_n]\) to the required one.

If, on the other hand, the synchronization of “positions” is the ultimate objective, then the \((n-1)\) identical \(H(s)\)-controllers must be of “PI-type”, i.e., \(H(s) = A_0 + A_1/1/s\). As already discussed, if \(A_1 < (a_{min})A_0\) then the MIMO system would be stable, the steady-state positions of the “Slave Plants” would
be "identical" with the steady-state position of the "Master Plant" and all the identical steady-state speeds would, again, have the value of \([K_n/a_n]\).

### 3.4 Ramp response \([x(t)=t]\)

Let

\[
\begin{align*}
X(s) &= \frac{1}{s^2} \\
H(s) &= A_0 + A_1 \frac{1}{s} + A_2 \frac{1}{s^2} \quad \text{(unit ramp input, } q = 1) \tag{15},
\end{align*}
\]

substituting Eqs. (15) and (16) in Eq. (4), we have

\[
E^{(n)}_{in}(s) = \left\{ \frac{[K_i-K_n]s+[K_i a_n-K_n a_i]}{s+a_n}D_1(s) \right\} + \left\{ \frac{s^2[s+a_i]\beta_i-s^2[s+a_n]\beta_n}{D_1(s)[a_i-a_n]+s^2[s+a_n]\beta_i-s^2[s+a_n]\beta_n} \right\},
\]

where

\[
D_1(s) = s^4 + a_i s^3 + K_i A_0 s^2 + K_i A_1 s + K_i A_2,
\]

\[
i = 1, 2, \cdots, n-1 \tag{17}
\]

or

\[
D_1(s) = s_4 + p_3 s^3 + p_2 s^2 + p_1 s + p_0. \tag{18}
\]

Applying Routh-Hurwitz stability criterion to Eq. (18), we get (Fortmann and Hitz, 1977):

\[
p_1[p_2 p_3-p_1] > p_0 p_3^2. \tag{19}
\]

Substituting values of \(p_3, \cdots, p_0\) in inequality (19), we must have

\[
K_i A_1 [(a_i) A_0 - A_1] > (a_i)^2 A_2. \tag{20}
\]

Inequality (20) is satisfied, if

\[
A_1 < (a_i) A_0, \tag{21}
\]

\[
A_2 < K_i \left[ \frac{A_0 A_1}{a_i} - \frac{A_1^2}{a_i^2} \right], \quad i = 1, 2, \cdots, n-1. \tag{22}
\]

Like for PI-controllers discussed in Sec. 3.3, inequality (21) is satisfied, if:

\[
A_1 < (a_{\min}) A_0, \tag{23}
\]

where, as before

\[
a_{\min} = \min\{a_1, a_2, \cdots, a_{n-1}\}.
\]

Therefore, in relation to \(A_0, A_1\), must be lie in the admissible region shown in Fig. 2. Our next task is to find, in relation to \(A_0\), the admissible region for \(A_2\).
For one particular value of \( a \), and a fixed value of \( A_0 \), Fig. 3 shows the plot of

\[
K_i \left[ \frac{A_0 A_1}{a_i} - \frac{A_i^2}{a_i^2} \right]
\]

versus \( A_1 \). Therefore, inequality (22) is satisfied, if

\[
A_2 < (K_i) \frac{A_i^2}{4}, \quad i = 1, 2, \ldots, n-1
\]

or

\[
\begin{align*}
A_2 < (K_1) \frac{A_1^2}{4} \\
A_2 < (K_2) \frac{A_2^2}{4} \\
\vdots \\
A_2 < (K_{n-1}) \frac{A_{n-1}^2}{4}
\end{align*}
\]

\[
\Rightarrow A_2 < (K_{\min}) \frac{A_2^2}{4},
\]

(24)

where

\[
K_{\min} = \min \{ K_1, K_2, \ldots, K_{n-1} \}.
\]

Figure 4 shows the admissible region for \( A_2 \) in relation to \( A_0 \).

**Note:**

As clearly stated in Sec. 2, \( K_{\min} \) and \( a_{\min} \) may belong to two different "Slave Plants" (S).

Therefore, if, for a "particular value" of \( A_0 \), inequalities (23) and (24) are satisfied, then all the \( 4(n-1) \) roots of Eq. (17) will fall in the left-half of \( s \)-plane, and, as in all the previous cases, we will have

\[
e_i^{(n)}(t) \bigg|_{t=\infty} = e_i^{(n)}(t) + e_i^{(n)}(t) \bigg|_{t=\infty} = 0 + 0
\]

\[
\Rightarrow y_j^{(n)}(t) \bigg|_{t=\infty} = y_j^{(n)}(t) \bigg|_{t=\infty}, \quad i = 1, 2, \ldots, n-1.
\]

Again, from Eq. (2), we have

\[
y_j^{(n)}(t) \bigg|_{t=\infty} = y_j^{(n)}(t) \bigg|_{t=\infty}
\]

\[
= \frac{1}{2!} \frac{d^2}{ds^2} \left[ \frac{K_n}{s+a_n} \exp(st) \right]_{s=0} + f^{(n)}(a_n, \beta_n, a_n)
\]

\[
= \frac{1}{2} \frac{K_n}{a_n} t^2 + f_1^{(n)}(t, K_n, a_n) + f^{(n)}(a_n, \beta_n, a_n)
\]

\[
= \frac{K_n}{a_n} \int [t] dt + f_1^{(n)}(t, K_n, a_n) + f^{(n)}(a_n, \beta_n, a_n),
\]

\[
i = 1, 2, \ldots, n-1
\]

where
"n" integral-plus-time constant plants

Fig. 3. Admissible region for parameter $A_2$ in relation to the parameter $A_1$ for a fixed value of $A_0$ and any one of the time constants $a_i$, $i = 1, 2, \ldots, n-1$ (for the PII controllers: $H(s) = A_0 + A_1(1/s) + A_2(1/s^2)$).

Fig. 4. Admissible region for parameter $A_2$ in relation to the parameter $A_0$, (for the PII controllers: $H(s) = A_0 + A_1(1/s) + A_2(1/s^2)$).

$$f^{(n)}_1(t, K_n, a_n) = -\frac{K_n}{a_n^2} t + \frac{K_n}{a_n^3}, \quad (25)$$

So,

- The MIMO system is stable \( \forall n, K_i, K_n, a_i, a_n, A_0, A_1 < (a_{min})A_0, A_2 < (K_{min})A_0^2/4 \) \((i = 1, 2, \ldots, n-1)\).
- \( y^{(n)}_1(t) = y^{(n)}_2(t) = \cdots = y^{(n)}_{n-1}(t) \big|_{t \to \infty} = y^{(n)}_n(t) \big|_{t \to \infty} = \frac{K_n}{a_n} \int [t] dt + \hat{f}^{(n)}_1(t, K_n, a_n) + f^{(n)}(a_n, \beta_n, a_n) \)

\( \Rightarrow \) fulfilment of Eqs. (A) and (B).

4. General Polynomial Input \[
\begin{bmatrix} x(t) = \sum_{h=0}^{q} \frac{1}{h!} t^h \end{bmatrix}
\]

The required controller $H(s)$ for $x(t) = u(t)$ and $x(t) = t$, discussed, respectively, in Secs. 3.3 and 3.4, will direct us that if the identical inputs $x(t)$ is of the form,
\[ x(t) = \frac{1}{q^i} t^q \quad q = 0, 1, 2, \cdots \]  
(26)

or

\[ x(t) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k, \]  
(27)

then, for either case, the controller \( H(s) \) must be of the type,

\[ H(t) = \sum_{k=0}^{\infty} A_k \frac{1}{s^k}. \]  
(28)

Having chosen the \( "q+2" \) coefficients of \( H(s) \) to meet the stability requirements, then for (26) and (27), we shall, respectively, have

\[ y_i^{(n)}(t) \big|_{t \to \infty} = y_i^{(n)}(t) \big|_{t \to \infty} \]

\[ = \frac{K_n}{a_n} \int \left[ \frac{1}{q^i} t^q \right] dt + f^{(n)}(t, K_n, a_n) + f^{(n)}(\alpha_n, \beta_n, \alpha_n), \]

\[ i = 1, 2, \ldots, n-1, \]  
(29)

\[ y_i^{(n)}(t) \big|_{t \to \infty} = y_i^{(n)}(t) \big|_{t \to \infty} \]

\[ = \frac{K_n}{a_n} \int \left[ \sum_{h=0}^{q} \frac{1}{h!} t^h \right] dt + \sum_{h=0}^{q} \int f^{(n)}(t, K_n, a_n) + f^{(n)}(\alpha_n, \beta_n, \alpha_n), \]

\[ i = 1, 2, \ldots, n-1, \]  
(30)

where

\[ f^{(n)}(t, K_n, a_n) = \frac{K_n}{a_n} (\frac{1}{q^i} t^q - \sum_{m=0}^{q} (-1)^m \frac{1}{m!} \frac{1}{a_n^{q+1-m}} t^m). \]  
(31)

(see Unbehauen and Vakilzadeh, 1989 b).

**Example 1.** Figure 5 shows the block diagram of a “3-plant” system with

\[ x(t) = 0, \quad H(s) = 1, \quad \text{(i.e., } A_0 = 1). \]

\[ S_1: \begin{align*} K_1 &= 1 \\ a_1 &= 0.2 \\ \alpha_1 &= -10 \end{align*} \quad S_2: \begin{align*} K_2 &= 2 \\ a_2 &= 0.3 \\ \alpha_2 &= 0 \end{align*} \quad \text{M:} \begin{align*} K_3 &= 3 \\ a_3 &= 1.5 \\ \alpha_3 &= 10 \end{align*} \]

**Discussion.** This is case (i), discussed in Sec. 3.1. As see in Fig. 6,

\[ y_1^{(3)}(t) = y_2^{(3)}(t) \big|_{t \to \infty} = y_3^{(3)}(t) \big|_{t \to \infty} = f^{(3)}(\alpha_3, \beta_3, \alpha_3) = \alpha_3 + \frac{1}{a_3} \beta_3. \]

\[ = 10 + 0 = 10. \]

**Example 2.** Figure 7 shows the block diagram of a “3-plant” system with
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\[
\alpha_1 = -10 \\
\beta_1 = 0 \\
G_1(s) = \frac{1}{s(s+0.2)} \\
G_2(s) = \frac{2}{s(s+0.3)} \\
G_3(s) = \frac{3}{s(s+1.5)}
\]

Fig. 5. Block diagram of Example 1.

\[
x(t) = 0 \\
H(s) = 1
\]

Fig. 6. Analogue computer response of Fig. 5.

\[
\alpha_1 = -10 \\
\beta_1 = 0 \\
G_1(s) = \frac{1}{s(s+0.4)} \\
G_2(s) = \frac{2}{s(s+0.3)} \\
G_3(s) = \frac{3}{s(s+1.5)}
\]

Fig. 7. Block diagram of Example 2.
\[ x(t) = u(t), \quad H(s) = 1 + 0.1 \frac{1}{s}, \quad \text{(i.e., } A_0 = 1, A_1 = 0.1) \]

\[
S_1: \begin{cases}
K_1 = 1 \\
da_1 = 0.4 \\
a_1 = -10 \\
\beta_1 = 0
\end{cases} \quad S_2: \begin{cases}
K_2 = 2 \\
a_2 = 0.3 \\
a_2 = 0 \\
\beta_2 = 0
\end{cases} \quad M: \begin{cases}
K_3 = 3 \\
a_3 = 1.5 \\
a_3 = 10 \\
\beta_3 = 0
\end{cases}
\]

**Discussion.** This is case (iii), i.e., step input and PI-controller, which was discussed in Sec. 3.3. Now,

\[
a_{\text{min}} = \min[a_1, a_2]
\]

\[
= \min[0.4, 0.3]
\]

\[
= 0.3,
\]

\[(a_{\text{min}})A_0 = 0.3\]

\[A_1 = 0.1 < (a_{\text{min}})A_0 = 0.3.\]

So inequality (12) is satisfied, and hence the MIMO system is stable. Therefore,

\[
y_1^{(3)}(t) = y_2^{(3)}(t)\bigg|_{t \to \infty} = y_3^{(3)}(t)\bigg|_{t \to \infty}
\]

\[
= \frac{K_3}{a_3} \int u(t) dt + \int_0^t f_1^{(3)}(t, K_3, a_3) + f_2^{(3)}(\alpha_3, \beta_3, a_3)
\]

\[
= \frac{3}{1.5} t + \left[ -\frac{K_3}{a_3^2} \right] + \left[ a_3 + \frac{1}{a_3} \beta_3 \right]
\]

\[
= 2t + \left[ -\frac{3}{(1.5)^2} \right] + [10]
\]

\[
= 2t + 8.7.
\]

That is, the three "identical" steady-state outputs are ramp function with slope: \((K_3/a_3) = 2\), which is vividly seen in Fig. 8.
Now, if plant (1) had been chosen to act as "Master Plant", then the three identical steady-state outputs would have been, again, ramp function with slope: \((K_1/a_1) = 2.5\); while if plant (2) had been selected to act as "Master Plant", then the three identical steady-state outputs would have been ramp function with slope: \((K_2/a_2) = 6.6\). As has been fully discussed before, if these plants were to be field-controlled dc-motors, then, with field-voltages of "one" volt, the three "position-synchronized" motors would have speeds of

- 2.5 [rad/sec] [for plant (1) as "Master Plant"]
- 6.6 [rad/sec] [for plant (2) as "Master Plant"]
- 2.0 [rad/sec] [for plant (3) as "Master Plant"]

in each case, of course, with appropriate coefficients for the required PI-controlers.

5. Conclusion

In this paper, synchronization of \(n\) integral-plus-time constant plants with non-identical gain and time constants, non-identical outputs initial conditions was studied – with two main objectives: (i) Making all the \(n\) steady-state outputs identical. (ii) Having these identical steady-state outputs time integral of the common inputs \(x(t)\). In a completely new way, it was shown how it would be possible to select any one of these \(n\) plants to act as "Master Plant" for the remaining \((n-1)\) plants, so that both of our objectives are fulfilled. In our to-day's world (where there is always restrictions on the available fund!) this implies that whenever a number of such similar machines are to be commissioned, then "one" reliable, and naturally expensive, machine can be procured to act as "Master Plant", while the remaining \((n-1)\) machines can be of less expensive type.

References


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"n" integral-plus-time constant plants

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