Robust Schur Stability of Interval Polynomials

F. Kraus, M. Mansour, and E. I. Jury

Abstract—In this note, different methods for the investigation of Schur stability of interval polynomials are considered and compared to each other.

I. INTRODUCTION

Since the results of Kharitonov [1] where he proved the robust Hurwitz stability of a family of interval polynomials

\[ f(s,a) = \sum_{i=0}^{n} a_i s^i \]  

several researchers looked for a corresponding result for Schur stability. Different counterexamples [2], [3] were found. In a series of papers [4]–[6] other conditions were looked for in order to prove the robust Schur stability for a continuum of polynomial parameters.

Using an assumption about the zero location of the vertex polynomials a Kharitonov-like result is given for Schur stability in [13]. Some results were found for a different parameter box [5], [7]. Other papers consider a polytopic region in the coefficient space and give more general results [12].

In this note, we describe different methods for the investigation of Schur stability using a Kharitonov parameter box. These methods are compared and discussed with each other. All these methods use a two-step algorithm. In the first step, the critical exposed edges are determined. In the second step, these edges are investigated with respect to stability. The various methods differ only in the way the stability of the edges is tested. Due to the specific Kharitonov box, the numerical load is kept within reasonable limits.

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II. MATHEMATICAL TOOLS

In the following, we collect the different important tools which can be used directly to solve the robust stability problem.

For \( f(z,a) \) with \( a \) in a polytopic region \( M \), a simplified version of the edge theorem [8] is given as follows.

**Theorem 1:** The family of polynomials \( f(z,a), a \in M \), robustly stable iff all the exposed edges of the polytope \( M \) are robustly stable.

Stability of the corners implies the stability of a subset of the edges. This follows from [3].

**Theorem 2:** Let \( f(z,a) \) be Schur stable on all corners of the parameter box (2) with

\[ a_i \in \{g_i, \bar{a}_i\}, \quad i = n/2 \mp \ldots, n \]

and \( a_i \) is constant elsewhere. Then \( f(z,a) \) is Schur stable for all edges, \( i = n/2 \mp \ldots, n \).

The problem of robust stability of generally directed edges is reduced to an eigenvalue problem [9].

**Theorem 3:** Let \( f_1(z) \) and \( f_2(z) \) be Schur stable. Then the polynomial family \( f(z, \lambda) = (1 - \lambda f_1(z) + \lambda f_2(z) \lambda \in [0, 1] \) is robustly stable iff the real eigenvalues of \( \Delta f_1 f_2 \) are positive. Here \( \Delta \) is the inner matrix for \( f(z), i = 1, 2 \).

In [10], edge stability is reduced to determining the number of zeros of an auxiliary polynomial \( G(z) \) on the stability boundary.

**Theorem 4:** Let \( f_1(z) \) and \( f_2(z) \) be Schur stable. Then the polynomial family \( f(z, \lambda) = (1 - \lambda f_1(z) + \lambda f_2(z) \lambda \in [0, 1] \) is robustly stable iff \( f_1(z^*) f_2(z^*) > 0 \) for every root \( z^* \) of \( G(z) = z^*[f_1(z^{-1}) f_2(z) - f_1(z) f_2(z^{-1})] \) with \( |z^*| = 1 \).

The conditions for Schur stability for low-order polynomials were directly investigated in [6]. It was shown that the complete stability test should be done only once. The rest of the tests deal only with the critical stability conditions [11]

\[ a_0 + a_1 + a_2 + a_3 + \ldots + a_n > 0 \]

\[ g_0 - \bar{a}_1 + a_2 - \bar{a}_3 + \ldots > 0 \]

and

\[ \det (\Delta(a)) > 0 \]

where \( \Delta(a) \) is the inner matrix for the polynomial \( f(z,a) \).

III. METHODS OF SOLUTION

A. Exposed Edges

We mention first the choice of the exposed edges which is common to all methods discussed hereafter. For the robust Schur stability with respect to the Kharitonov parameter box, according to the edge theorem [8], it is necessary and sufficient to consider the robust stability of all the edges. But with [3] only the \( a_i \)-edges which result from the variation of \( a_i \), \( i = 0, \ldots, (n - 1)/2 \) should be checked for stability for all possible combinations of

\[ a_i = \{g_i, \bar{a}_i\}, \quad i = n/2 \mp \ldots, n \]

Every combination of \( a_i \) in (5) characterizes a group of edges. According to the procedure in [6], one checks first the linear necessary conditions (3), then the stability for one of the corners of each group of edges. Finally, for all \( a_i \)-edges of one group it is sufficient to check only the critical stability condition (4). This is a positivity test of a polynomial in the variables \( a_0, \ldots, a_n \). Thereby

\[ x \uparrow \times i \]

denote the next higher integer or next lower one to \( x \), respectively.
for every \( a_i \)-edge we obtain the critical edge polynomial in a single variable. Its degree is \( a_i \) or \( a_{i-1} \), \( i = 0, \ldots, (n - 1)/2 \). If \( n - 1 - i \), the methods which are described in the following differ only in the way how the edge test is made.

**B. Direct Test for the Stability of the Edges**

This method is described in [6] and is based on the explicit evaluation of the critical edge polynomial for the \( a_i \)-edge given by

\[
E_p(a_i) = \det \left( \Delta(a) \right)
\]

(6)

where \( \Delta(a) \) is Jury's inner matrix for the polynomial \( f(z, a) \). For stability along the \( a_i \)-edge, the critical edge polynomial \( E_p(a_i) \) should remain positive for \( a_i \in [g_i, \bar{a}_i] \). This positivity test of the critical edge polynomial \( E_p(a_i) \) can be executed directly by a numerical determination of zeros of \( E_p(a_i) \) or indirectly using Sturm theorem to check the existence of real roots for \( a_i \) in \([g_i, \bar{a}_i] \).

In simple cases, e.g., when \( E_p(a_i) \) is a straight line or a parabola, simple conditions will result [6].

Although the expressions for \( \det(\Delta) \), its derivative with respect to \( a_i \) and the ordering according to the powers of \( a_i \) can be obtained using a program for symbolic processing like MATHEMATICA, the computation time increases rapidly with \( n \). It is to be noted that this is done only once.

An alternative method is the numerical determination of the critical edge polynomial. The \( a_i \)-edge polynomial is given by

\[
E_p(a_i) = \gamma_0 + \gamma_1 a_i(\alpha) + \gamma_2 a_i^2(\alpha) + \cdots + \gamma_n a_i^n(\alpha)
\]

(7)

where

\[
a_i(\alpha) = g_i + \alpha (\bar{a}_i - g_i), \quad \alpha \in [0, 1]
\]

\[
\nu = \max \{ n - j; j \} - 1.
\]

If there does not exist a real eigenvalue \( \lambda \) for \( \Delta(a^*) \) with \( a^* \) nonsingular, we get

\[
\delta_j \Delta^{-1}(a^*) B_j x = \lambda x
\]

(13)

where \( \lambda = -1/\alpha \). If no real eigenvalues with \( \lambda \geq -1 \) exist, then the corresponding edge is stable.

The eigenvalue problem (11) for the parameter box (2) can be simplified because of the special structure of the matrix \( B_j \)

\[
\begin{bmatrix}
0 \\
1 \\
\end{bmatrix}
\]

(14)

where the nonzero columns of \( B_j \) are included in \( \bar{B}_j \). Then from (13)

\[
0 \delta_j \Delta^{-1}(a^*) \bar{B}_j x = \lambda x
\]

(15)

which can be written in the form

\[
\begin{bmatrix}
0 [A_2] [x_1] \\
0 [A_1] [x_2]
\end{bmatrix} = \lambda [x_2]
\]

(16)

the reduced eigenvalue problem is obtained

\[
A_2 x_2 = \lambda x_2.
\]

(17)

The order of the reduced eigenvalue problem for the significant \( a_i \)-edges is \( \sum_{i=0}^{(n-1)/2} i \). Note that the whole matrix \( \Delta(a^*) \) must be inverted.

Another possibility to simplify the eigenvalue problem is to use the transformation matrix

\[
T = \begin{pmatrix}
\bar{B}_j \\
(\bar{B}_j \bar{B}_j)^{-1} \bar{B}_j
\end{pmatrix}
\]

(18)

where \( \bar{B}_j \) is the column complement of \( \bar{B}_j \). The original generalized eigenvalue problem (12) is transformed to

\[
T \begin{pmatrix}
\alpha B_j + \frac{1}{\delta_j} \Delta(a^*) \\
\end{pmatrix} x = 0
\]

which yields the reduced eigenvalue problem

\[
(A_{11} A_{12} - A_{21} A_{22}) x_2 = 0
\]

(19)

**D. Edge Stability Using Colinearity Conditions**

Assuming the stability of \( f(s) \), then for the stability of the family of polynomials

\[
f(s, \lambda) = (1 - \lambda) f(s) + \lambda f_a(s)
\]

(20)

along the edge \( \lambda \in [0, 1] \), it is sufficient to show that \( f(s, \lambda) \) does not have a zero \( s = a^* \) on the stability boundary for all \( \lambda \in [0, 1] \). The polynomial \( f(s, \lambda) \) vanishes for the \( s = a^* \) iff the two vectors

\[
f(s) = \begin{pmatrix}
\text{Re} f(s) \\
\text{Im} f(s)
\end{pmatrix}
\]

(21)

are colinear and in an opposite directions.

For all points \( s = a^* \) on the stability boundary where the colinearity is valid, one must check the direction. If the vectors \( f(s) \) and \( f_{s}(s) \) are opposite, i.e., \( f(s) = -k f_{s}(s) \) for some real positive \( k \), instability occurs. This is a new formulation of the idea of the method in [10].
The colinearity condition can be formulated as
\[ \det \left[ f_1(s^*), f_2(s^*) \right] = 0. \]  
(22)
For colinear vectors \( f_1(s^*) \) and \( f_2(s^*) \) the orientation can be checked by the scalar product \( f_1(s^*)^T f_2(s^*) \). If the scalar product is negative then \( f(s^*) \) vanishes on the edge between \( f_1(s^*) \) and \( f_2(s^*) \) and instability occurs. For special stability regions, the colinearity test can be simplified.

For Schur stability it is simpler to decompose \( f(z) \) using the symmetrical and antisymmetrical parts of \( f(z) \)
\[ f(z) = f(z) + g(z) \]  
(23)
where
\[ h(z) = \left[ f(z) + z^n f(z^{-1}) \right]/2 \]
\[ g(z) = \left[ f(z) - z^n f(z^{-1}) \right]/2. \]  
(24)
The symmetrical and antisymmetrical parts are mutually orthogonal on the unit circle. Therefore, for colinearity we get the determinant condition
\[ G(z) = \det \left[ h_1(z), h_2(z) \right] \]
\[ = h_1(z) g_2(z) - g_1(z) h_2(z) = 0. \]  
(25)
This polynomial condition can be transformed to a condition on the original polynomials \( f_1(z) \) and \( f_2(z) \) using the relation (24):
\[ G(z) = z^n \left[ f_1(z^{-1}) f_2(z) - f_1(z) f_2(z^{-1}) \right]/2. \]  
(26)
The polynomial to be checked \( G(z) \) is in general of degree 2. For the special case of an \( a_k \)-edge of Kharitonov parameter box, the degree of resulting polynomial \( G(z) \) however reduces to 2\( (n-v) \) where \( v = \min \{j, n-j\} \).

The polynomial \( G(z) \) is even and antisymmetrical. As it was shown in [7] it is possible to eliminate the 2 zeros at \( \pm 1 \), i.e., the factor \( z^2 - 1 \). The resulting polynomial
\[ G'(z) = \prod_{i=1}^{n-v-1} \left( z + \sigma_i \right) \]  
(27)
can be obtained by the projection transformation without any calculation of zeros and is of degree \( n-v-1 \). Therefore the same complexity of the solutions as (17), (19) will be reached. In other words, instead of the eigenvalue problem of order \( n-v-1 \) we need to examine if roots of a polynomial of degree \( n-v-1 \) are in the interval \([-1,1]\).

Application of Sturm theorem is also possible. If \( G'(z) \) has roots in the interval \([-1,1]\) it is nevertheless necessary to determine these roots numerically. For every one of these roots \( \sigma \) we have to check the second part of the stability condition, i.e.,
\[ \frac{f_1(\sigma i + i\sqrt{1 - \sigma^2})}{f_1(\sigma i + i\sqrt{1 - \sigma^2})} > 0. \]  
(28)
For the \( a_k \)-edge we have with \( f_2(z) = f_1(z) + \delta_k z^{n-k} \) the condition
\[ 1 + \delta_k z^{-k} \frac{z^{n-k}}{f_1(z)} |_{z=\sigma i + i\sqrt{1 - \sigma^2}} > 0. \]  
(29)

IV. Conclusions

In this note, the discrete counterpart of Kharitonov's theorem is obtained. The solution is based on the use of Hollot-Bartlett-Huang theorem [8] and Hollot-Bartlett theorem [3]. This enabled us to test for Schur stability only a subset of the edges.

The Schur testing of the required edges of the cube is performed using three different methods, namely, the critical edge polynomial, edge stability as an eigenvalue problem, and edge stability using colinearity conditions. Comparison of these three methods is discussed. It is believed that with the Schur testing of the minimum number of edges and the use of the critical stability constraints the minimum computational effort can be achieved.

REFERENCES

Modal Control of Large Flexible Space Structures
Using Collocated Actuators and Sensors

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Abstract—This note considers the problem of assigning the eigenvalues associated with critical modes of a large flexible space structure

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